A TREATISE

ON

GEOMETRICAL OPTICS.

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ON

GEOMETRICAL OPTICS

BY

Robert Samuel
R. S. HEATH, M.A., D.Sc.,

FELLOW OF TRINITY COLLEGE, CAMBRIDGE,
PROFESSOR OF MATHEMATICS IN THE MASON COLLEGE, BIRMINGHAM.

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PREFACE.

In the following work an attempt has been made to give as complete an account of modern Optics, including the labours of Gauss, Listing, Maxwell, Helmholtz and Abbé, as could be compressed within the limits of a single volume; at the same time much that is of interest and importance, both theoretical and practical, has been necessarily omitted. The subjects have been treated as far as possible in a natural order, beginning with the simplest and advancing to the most general and complex. Thus the reflexion and refraction of single rays of light are considered before the corresponding properties of pencils, and the complete approximate theory of lenses is given before the theory of caustics and aberrations, and these before the general theory of thin pencils. The detailed consideration of heterogeneous media, which may be said to fall outside the province of optics, has been postponed to the last chapter. Gauss' theory of lenses has been worked out completely by elementary geometrical methods, so as to bring it within the reach of all students; while, at the risk of some repetition, his own elegant analysis has been given in a separate chapter. Before treating the refraction of thin optical pencils after the manner of Maxwell, a short account of the general properties of all thin pencils which are not systems of normals, has been introduced. I append a list of the most important memoirs and treatises relating to this subject, most

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of which I have been able to consult. The list has no pretence to completeness; for numbers of memoirs have been so often incorporated into text-books that it is not necessary to specify them in particular. I would mention Lloyd's "Treatise on Light and Vision," a most valuable though scarce work, as having been of special service; the chapter on spherical aberration and the general description of the instruments have been derived mainly from this treatise. Cayley's Memoirs on Caustics, Maxwell's papers on Thin Pencils, Helmholtz' Physiologische Optik and Abbé's papers on the microscope, have formed the basis of the sections dealing with these various subjects.

Any notifications of inaccuracies or suggestions which may add to the usefulness of the work will be most gratefully received.

R. S. HEATH.

MASON COLLEGE, BIRMINGHAM.
April, 1887.

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CHAPTER I.

THE NATURE AND GENERAL PROPERTIES OF LIGHT.

1. LIGHT may be defined as the external conditions which, acting through the instrumentality of the eyes, produce in the brain the sensation called sight. At various periods in the history of the science of Optics different and sometimes conflicting theories have been advanced to explain the nature of light. It is now usually supposed that light consists of vibrations of a highly elastic solid medium pervading all space, which has been provisionally called the æther. But, though opinions respecting the nature of light have been divided, there are a few of its leading properties which have been amply established by experience and are universally recognised as fundamental and dependent upon no hypothesis whatever; and any theory on the nature of light has to furnish a satisfactory explanation of these properties before it can be accepted. The province of Geometrical Optics is to deduce by the methods of geometry the consequences of these general properties and thereby to explain the less obvious modifications which light undergoes, and to apply them to the construction of instruments for the improvement of our sight and the examination of objects too minute or too distant to be seen distinctly by the naked eye.

2. Any space through which light can pass, whether it be occupied by matter or not, is called a *medium*. In any homo-

geneous medium light travels with uniform velocity in straight lines.

Light consists of separable and independent parts. If part of the light proceeding from a luminous body be intercepted by an opaque obstacle, this will not in any way affect the remaining portion which is allowed to pass. Also, in general, light from two independent sources may travel along the same path without interference. These two experimental facts show that light is capable of a quantitative measurement. For the present we shall suppose that the light with which we are dealing is all of the same kind and homogeneous, and that its quantity or intensity is measured in terms of some fixed standard.

When light travels through any medium of which we have cognisance, part of it is absorbed by the medium, and only part of it is transmitted. But in what follows, unless the contrary be stated, the media will be supposed to be perfectly transparent, that is, they will transmit the whole of the light incident upon them.

It will often be convenient to consider the portion of light which travels along some particular line in the medium apart from the rest; such a portion of light is called a *ray*, and it will be supposed to have the form of an indefinitely slender cone, whose axis is the line in question. A collection of rays which during their course never deviate far from some fixed central ray is called a *pencil* of rays, and the fixed central ray is called the *axis* of the pencil. If the rays of a pencil meet in a point, that point is called the *focus* of the pencil.

As we shall have continually to mention the eye in the course of subsequent investigations, it may be well here to refer very briefly to the theory of the eye; the details of the theory cannot be given till later. The pencil of rays proceeding from a point and limited by the aperture of the pupil, is by the crystalline lens of the eye brought to a focus on the retina, and the point is seen by means of such an image on the retina. Each point of a surface gives a corresponding image, and thus we are enabled to form a mental picture of a surface.

3. Observation leads us to distinguish certain bodies, which may be called self-luminous, whose presence is necessary to excite our organs of sight. Bodies which in themselves are not luminous

become luminous in the presence of a self-luminous body and are then visible to us. This distinction is immaterial for our present purpose however; in treating of the emission of light from a body it will not be necessary to consider whether the body is self-luminous or luminous through the presence of other bodies; the laws of emission are the same in both cases.

Let dQ be the quantity of light emitted by a bright point or an indefinitely small element of a bright surface, within a small cone of solid angle $d\omega$, whose vertex is at the origin of light and whose axis is in a given direction, then the intensity of the emission of light in that direction may be measured by $\frac{dQ}{d\omega}$.

A bright body emits light in all directions, but the intensity of emission is different for different directions. The law of emission is given by a well-known experiment. Luminous bodies appear of the same brightness whatever be the inclination of the bright surface to the line of sight. Thus if a cylinder of silver be heated till it becomes luminous and taken into a dark room, it cannot be distinguished from a perfectly flat bar; and similarly, a luminous sphere (like the sun as seen through a mist) appears like a flat disc. The same experiment is true of the intensity of the heat rays radiated from a hot body; in this form it is intimately associated with the Theory of Exchanges.

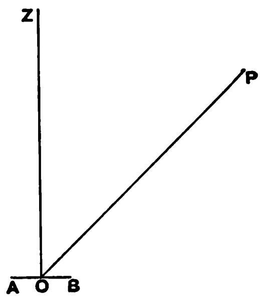
This experiment shows that *the intensity of emission of light from any element of a bright surface in any direction is proportional to the cosine of the inclination of the direction of emission to the normal to the element of the surface.*

For suppose that a bright body is viewed through a tube of small aperture; when the tube is directed so that the element of the bright surface seen is normal to the line of sight, let the area of the element be ω. Then when the tube is directed so that the normal to the element of the bright surface seen through the tube makes an angle θ with the line of sight, the area of the element will be $\omega \sec\theta$. Let $f(\theta)$ be the intensity of emission per unit area in a direction making an angle θ with the normal to the element; then the whole amount of light transmitted to the eye when the element is inclined to the line of sight at an angle θ is $\omega \sec\theta \,.\, f(\theta)$.

But this, by experiment is independent of θ, and therefore

$$f(\theta) \propto \cos\theta.$$

4. Let dS be the area of an element of the bright surface and let μdS denote the intensity of the light emitted in the direction of the normal to the element. Then μ may be called the *intrinsic brightness* of the element.



Let AB be the element, OZ the normal to it, and let OP be a direction making an angle θ with the normal OZ, and such that the plane POZ makes an angle ϕ with a given fixed plane through OZ. Describe a small cone whose axis is OP, and let the solid angle of the cone be $d\omega$. Then the quantity of light emitted within this small cone is $\mu dS \cos\theta \, d\omega$, or $\mu dS \cos\theta \sin\theta \, d\theta \, d\phi$.

The whole quantity of light emitted by the element dS in all directions will therefore be

$$\mu dS \iint \sin\theta \cos\theta \, d\theta \, d\phi,$$

the limits of integration being

$$\left.\begin{array}{l}\phi = 0 \text{ to } \phi = 2\pi \\ \theta = 0 \text{ to } \theta = \frac{1}{2}\pi\end{array}\right\}.$$

This gives on integration,

$$\mu dS \, 2\pi \,.\, \tfrac{1}{2} \text{ or } \mu\pi dS.$$

Hence if we denote the whole quantity of light emitted by the element per unit area by μ', the intensity of emission per unit area in a direction making an angle θ with the normal will be $\mu'/\pi \,.\, \cos\theta$, and therefore the intrinsic brightness is μ'/π.

Similarly, if μ'' denote the quantity of light emitted in all directions from a luminous point, the intensity of emission in any direction will be $\mu''/2\pi$.

5. If dQ be the quantity of light which falls on a small area dA of an illuminated surface surrounding a given point of the surface, then $\frac{dQ}{dA}$ is called the *intensity of illumination* of the surface at that point.

We shall now find the illumination of a small area dA due to an element of any bright surface dS. Let O be the centre of the element of the luminous surface and C the centre of the illuminated area dA, and let $OC = r$. Let θ be the inclination of OC to the normal at O, and ϕ that of OC to the normal at C.

Then if dA subtend a solid angle $d\omega$ at O, the quantity of light it receives will be $\mu dS \cos\theta\, d\omega$, where μ is the intrinsic brightness of the element.

But
$$d\omega = \frac{dA \cos\phi}{r^2};$$

and therefore the quantity of light received by dA from the element dS is

$$\mu\, dS\, dA\, \frac{\cos\theta\cos\phi}{r^2}.$$

This is symmetrical with regard to the two elements, and would therefore represent the quantity of light received by dS from the element dA, were it of intrinsic brightness μ.

Let $d\sigma$ be the solid angle subtended at C by the bright element dS, so that

$$d\sigma = \frac{dS\cos\theta}{r^2};$$

intensity of

then the illumination of dA due to the element dS is

$$\mu d\sigma \cos\phi.$$

6. The illumination of a small area dA due to any finite surface of uniform brightness may now be found.

Take any small element of the bright surface about a centre O, and as before let $d\sigma$ be the solid angle subtended by it at the centre C of the illuminated area. Let ϕ be the angle between the line OC and the normal at O. Then the illumination due to the bright element is

$$dI = \mu\cos\phi\, d\sigma.$$

Describe a sphere of unit radius with C as centre; this will cut the small cone subtended at C by the element of area, in a section whose area is $d\sigma$; hence $d\sigma \cos\phi$ will be the projection of this small section on the plane of the illuminated area at C, and may be denoted by $d\varpi$; then

$$dI = \mu d\varpi.$$

By integration we arrive at the following method of determining the illumination of a small area at C due to any finite bright surface.

From C draw radii to all points of the boundary of the bright surface as seen from C, forming a conical surface. Let the part of the surface of the sphere of unit radius whose centre is C, intercepted within the cone, be projected on the plane of the area at C. If ϖ be the area of the projection, the illumination of the area will be given by the equation

$$I = \mu\varpi.$$

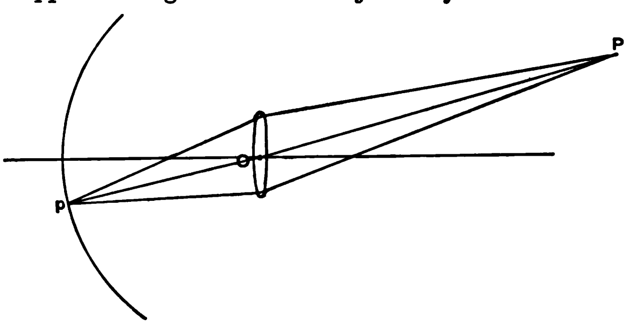
Ex. To find the illumination due to a spherical luminary.

Let a be the semi-vertical angle of the cone whose vertex is at the centre of the area and which envelopes the bright sphere. The curve in which this cone cuts the sphere of unit radius is a circle whose radius is $\sin a$. Hence if θ be the zenith distance of the luminary, the illumination on a small horizontal area is

$$I = \mu\pi \sin^2 a \cos\theta.$$

7. *Objects appear equally bright at all distances.*

The apparent brightness of an object may be measured by the



whole quantity of light entering the eye from the object divided by the area of the picture of the object on the retina of the eye.

Let P be any point of the object, p the corresponding point of the picture on the retina; then it will be shown afterwards that

the line Pp passes through the fixed point O, the optical centre of the eye.

Let S be the area of a small object and s that of the picture and let $OP = R$, $Op = r$. Then $S : R^2 = s : r^2$.

Now the quantity of light entering the eye is $\mu S\omega/R^2$, where ω is the area of the aperture of the eye. This may be written $\mu s\omega/r^2$; hence, dividing by s, we get the intrinsic brightness of the image, which is equal to $\mu\omega/r^2$.

We shall assume for the present that as the eye adjusts itself to different distances, r does not change.

Thus *the apparent brightness is independent of the distance of the object.*

The area of the aperture of the eye changes according to the brightness of the light. But if we suppose the aperture to remain the same, as the object is removed, no change in brightness has taken place, so that the aperture does not need further adjustment.

When the object is very distant the area of the picture on the eye gets to be very small indeed, so that the nerves of the retina cannot distinguish it from a point. In this case the brightness must be measured simply by the quantity of light; and therefore, by the same investigation, *the brightness varies inversely as R^2.*

EXAMPLES.

1. A small white surface being placed horizontally on a table, and illuminated by a lamp or candle placed at a given horizontal distance a, show that the height of the flame from the table which will give the greatest possible illumination is equal to $\frac{a}{\sqrt{2}}$.

2. If candles of equal brightness be placed at the angular points of a regular polygon, prove that a small plane area placed at the centre of the polygon will be equally bright on both sides, whatever be the orientation of its plane.

3. An elliptic arc bounded by the major axis $2a$ is illuminated by a bright point at the focus S; show that the illumination is a minimum at a point P such that $SP = \frac{4}{3}a$, if the eccentricity be greater than $\frac{1}{3}$. Investigate also the points of maximum and minimum illumination when e is less than $\frac{1}{3}$.

4. If q be the measure of the illumination at any point of a horizontal plane caused by the sky supposed uniformly bright, and q' the illumination where the brightness varies as the cosine of the zenith distance (the zenith brightness being the same in both cases), show that $q' = \frac{2}{3}q$.

5. A bright point is placed at a distance r from the centre of a sphere whose radius is a. Show that the average illumination of the surface of the sphere is

$$\frac{I}{a^2} \cdot \frac{r - \sqrt{r^2 - a^2}}{r - a}.$$

6. A curve is illuminated by a bright point in its own plane. If the illumination is the same at every point, show that the equation to the curve is $r^2 = a^2 \sin 2\theta$, or else a circle. In what relation do the two solutions stand to one another?

7. An infinitely long luminous vertical straight line stands on a horizontal table. Show that the illumination at a point on the table distant r from the foot of the luminous line varies inversely as r.

8. Find the form of a surface of revolution such that it may be uniformly illuminated by light proceeding from a point in its axis, and show that the illumination will still be uniform, if the surface scatter partially, but uniformly, the light incident on it.

9. A luminous point is placed on the axis of a truncated conical shell; prove that the whole illumination of the surface of the shell varies as

$$\frac{c_2}{(c_2^2 + a_2^2)^{\frac{1}{2}}} \pm \frac{c_1}{(c_1^2 + a_1^2)^{\frac{1}{2}}},$$

where a_1, a_2 are the radii of the circular ends of the shell and c_1, c_2 the distances of the luminous point from their plane.

10. A right cone of vertical angle 2θ is described about a given self-luminous sphere, and at the points of the sphere in which the axis of the cone cuts it, tangent planes are drawn; prove that the mean illumination of that part of the cone which is enclosed between these two planes varies as $\cos\frac{\theta}{2}\cos^2\theta$.

11. A right cone, the radius of whose base is to its height as $1 : \sqrt{2}$ stands on a table and its surface is uniformly self-luminous; show that the illumination on the table at a distance from the axis of the cone equal to its height, is

$$I\pi\left(\frac{1}{4} - \frac{1}{3\sqrt{3}}\right).$$

12. A uniformly bright isosceles triangle is placed with its plane vertical and its base on a horizontal table. Prove that the illumination at a point O of the table such that the line joining O to the middle point of the base

is perpendicular to the base and equal to half of it, varies as

$$\frac{\pi}{4} - \frac{a}{(2b^2 - a^2)^{\frac{1}{2}}} \cos^{-1} \frac{a}{b\sqrt{2}},$$

where $2a$ is the length of the base, and b that of either side.

13. A uniformly bright area, in the form of a quadrant of a circle, has one of its bounding radii in a given plane to which the plane of the quadrant is at right angles; prove that the curves of equal illumination on the given plane are found by eliminating ϕ between $r \sin \phi = a \cos(\theta - \phi)$, and $a^2 \cos^2 \theta e^{2(\phi - \gamma)\tan\theta} = (r^2 + a^2)\sin^2 \phi$.

14. Two non-intersecting surfaces wholly within view of one another are bounded by circles in parallel planes perpendicular to the straight line joining the centres of the circles. The distance c between the planes is large compared with the radii a and b of the circles. Show that, if one of the surfaces be uniformly luminous, the intensity of illumination being C per unit area, the total amount of light intercepted by the other surface is approximately

$$\pi C \frac{a^2 b^2}{c^2} \left\{ 1 - \frac{a^2 + b^2}{c^2} + \frac{a^4 + b^4 + 3a^2 b^2}{c^4} \right\}.$$

15. A bright sphere rolls rapidly on a table along a circle whose centre is A. Show that the curves of equal apparent illumination are concentric circles, the apparent illumination on any one being

$$\frac{2}{\pi} I' \int_0^{\frac{\pi}{2}} \frac{d\theta}{(1 - k^2 \sin^2 \theta)^{\frac{3}{2}}},$$

where $(1 - k^2)^{\frac{3}{2}} = \frac{I'}{I}$, and I, I' would be the illuminations at the points on the circle nearest to and furthest from the sphere if it were at rest.

CHAPTER II.

Reflexion and Refraction of rays of Light.

8. When a ray of light travelling in one medium is incident on the surface of another medium, it is usually divided into three separate portions which behave in different manners.

(i) A portion is reflected back into the original medium, in a direction determined according to a certain law.

(ii) Another portion passes into the new medium, having its direction changed according to another law; this portion is said to be *refracted* into the new medium.

(iii) A third portion is said to be *scattered* by the surface bounding the two media; the bounding surface becomes illuminated and itself acts like a source of light sending rays in all directions.

When a ray of light is incident on a solid opaque body, the second portion does not exist, and all the light is either reflected or scattered. The quantity of light reflected depends upon the nature of the surface; the smoother and more highly polished the surface is, the more light is reflected. The scattering of light is probably due to the unevenness of the surface; the incident light is reflected by minute portions of the surface which act as mirrors distributed irregularly in all directions. It is by the scattering of light that non-luminous bodies become visible when in the presence of a bright body.

9. The plane containing the incident ray and the normal to the surface separating the two media, is called *the plane of incidence*, and the acute angle between the incident ray and the normal is called *the angle of incidence*, and the acute angle between the reflected ray and the normal, *the angle of reflexion*.

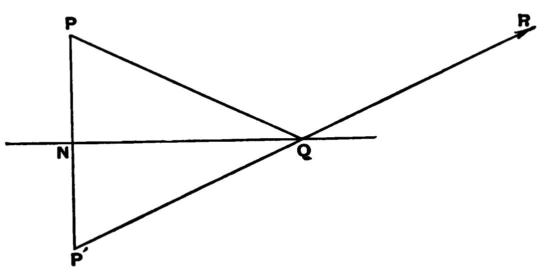
When the direction of a ray of light is changed by reflexions or refractions, the angle through which the original ray produced must be turned in order to bring it into the position of the final ray, is called the *deviation* of the ray.

The law according to which a ray of light is reflected at a surface may be thus stated.

The angles of incidence and reflexion always lie in the same plane and are equal to each other.

This is an experimental law which may be verified by direct observation. The most accurate direct verification of the law of reflexion is furnished by the use of the transit circle and a trough of mercury to determine the altitude of a star. The telescope is first directed towards the star itself and then towards the trough of mercury which is placed so that the star can be seen by reflexion at the surface of the mercury. The two readings are taken, and it is found that the difference of the readings is double of the altitude of the star. Since the surface of the mercury assumes the form of a horizontal plane under the action of gravity, and since the rays of light from a star are parallel, it easily follows that the ray incident on the mercury and the ray reflected from it make equal angles with the vertical. The measurements by the transit circle are of the utmost accuracy, and the law has always been found to be absolutely true to the degree of accuracy of which the instrument is capable.

10. If the ray be incident on a plane surface the reflected ray may be found by a simple geometrical construction. If P be any



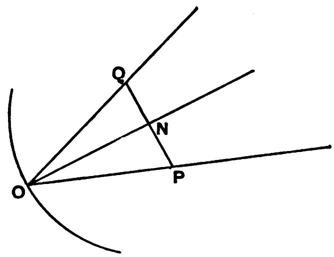
point on the incident ray PQ, and if from P a perpendicular PN be drawn to the reflecting plane and be produced to P' so that $P'N$

is equal to PN, then by elementary geometry it is clear that $P'Q$ produced is the direction of the reflected ray.

If the surface be not plane, we may substitute the tangent plane to the surface at Q, for the plane of the mirror in the previous construction.

11. This law of reflexion may be expressed by means of an analytical formula.

Let ON be the normal to the reflecting surface, PO and OQ



the incident and reflected rays respectively, and let ϕ be the angle of incidence. Measure OP and OQ each equal to unity, and join PQ, meeting the normal in N. Then by the law of reflexion, N is the middle point of PQ, and ON is perpendicular to PQ, so that

$$ON = \cos\phi.$$

Let (p, q, r) be the direction cosines of the normal, and (l, m, n), (l', m', n') the direction cosines of the incident and reflected rays, respectively, referred to any rectangular axes through O; then (l, m, n) are the coordinates of the point P, (l', m', n') the coordinates of Q. If we express the fact that N is the middle point of PQ, we find the equations

$$\left.\begin{aligned} l + l' &= 2\cos\phi \,.\, p \\ m + m' &= 2\cos\phi \,.\, q \\ n + n' &= 2\cos\phi \,.\, r \end{aligned}\right\}.$$

We may substitute the value of $\cos\phi$, in terms of the direction cosines, from either of the equations

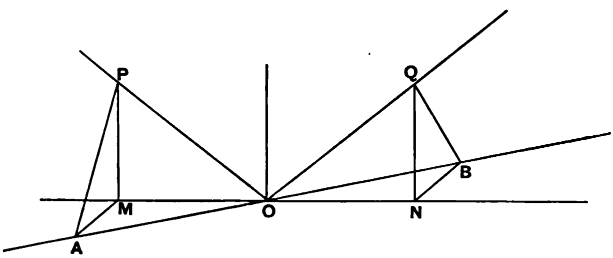
$$\cos\phi = lp + mq + nr,$$
$$\cos\phi = l'p + m'q + n'r;$$

and then the equations are linear equations in l, m, n, l', m', n'.

As soon as we know the direction cosines of the incident ray and the normal, the equations will give the direction cosines of the reflected ray. The equations are equivalent to two independent equations only; for if we multiply them, respectively, by p, q, r, and add the results, we get an identity. The values of l', m', n' as derived from the equations, will satisfy the relation

$$l'^2 + m'^2 + n'^2 = 1.$$

12. It may easily be proved by elementary geometry, that *when a ray is reflected at a plane surface the incident ray*



and the reflected ray make equal acute angles with any line in or parallel to the reflecting plane.

For let PO, OQ be the incident and reflected rays, and let the plane of incidence meet the reflecting plane in the line MON. Also let AOB be a line drawn through O parallel to the given line. On the lines OP, OQ measure equal lengths OP, OQ, and through P, Q draw planes perpendicular to the line MON meeting this line in the points M, N and the line AOB in the points A, B, respectively. Then since OP is equal to OQ, and the angle POM equal to the angle QON, it easily follows that $OM = ON$ and $PM = QN$; also that $OA = OB$, $MA = NB$; and therefore that the triangles PAM, QBN are equal in all respects. From these results, it follows that the triangles AOP, BOQ are equal in all respects, and therefore that the angles AOP, BOQ are equal. This proves the proposition.

Conversely, if two lines PO, OQ lie in a plane normal to the reflecting plane and make equal acute angles with any given line in the plane, they may be taken to represent an incident and reflected ray, respectively. The proof is similar to the preceding.

To prove these theorems analytically, we take the normal to

the reflecting plane at the point of incidence as the axis of z, so that, with the previous notation, $p=0$, $q=0$. The formulæ then show that

$$l+l'=0.$$

This is true for all directions of the axis of x and shows that the incident and reflected rays make supplementary angles with any line taken as the axis of x.

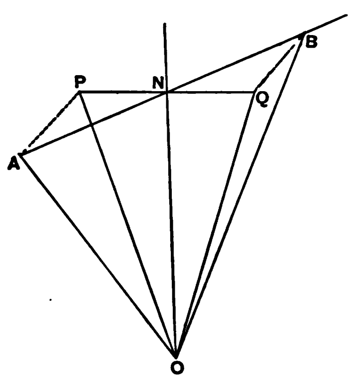
To prove the converse, we take the plane of incidence as the plane of xz, and suppose that the direction cosines of the given line are $(\lambda, \mu, 0)$. Then $m=0$, $m'=0$, and since PO, OQ make supplementary angles with the line $(\lambda, \mu, 0)$

$$\lambda l+\mu m+\lambda l'+\mu m'=0,$$

which is equivalent to $l+l'=0$. This proves that PO, OQ are related like an incident and a reflected ray.

It follows from the preceding proposition that *if a ray of light be reflected in any manner successively at two plane surfaces, the initial and final rays are equally inclined to the line of intersection of the plane surfaces.*

13. *If a ray of light be reflected at a surface, the projections of the incident and reflected rays on any plane through the normal, themselves obey the law of reflexion.*



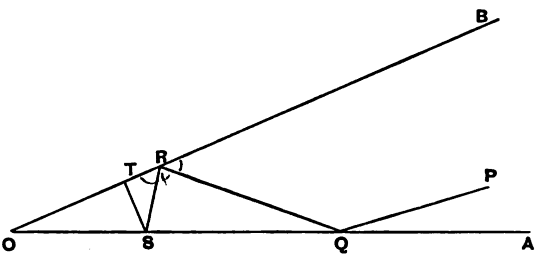
For along the incident and reflected rays measure equal distances OA, OB; then AB will be bisected at right angles by the normal to the surface ON. Let PNQ be the projection of ANB on any plane through the normal, so that OP, OQ are the

projections of the incident and reflected rays. Then it is clear that $PN = NQ$, and therefore OP, OQ are equally inclined to the normal ON.

Further, it is easily seen that the triangles AOP, BOQ are equal in all respects, and therefore the angle AOP is equal to the angle BOQ. In other words, *the incident and reflected rays are equally inclined to any plane through the normal.*

14. Let a ray of light be reflected successively at two plane mirrors, to find the direction of the ray after any number of reflexions.

We shall first consider the case in which the reflexions take place in a plane perpendicular to both mirrors.



Let OA, OB be the plane mirrors and let $PQRST$...be the ray of light which is reflected successively at Q, R, S, T...

Let ϵ denote the angle between the mirrors, and let θ_1, θ_2, θ_3... be the acute angles formed by the ray with the reflecting surfaces at the successive incidences. Thus the angles at Q are each θ_1, those at R, θ_2, and so on; so that from the triangle QOR we find $\theta_2 = \theta_1 + \epsilon$, and similarly $\theta_3 = \theta_2 + \epsilon$, &c. These equations may be written

$$\theta_2 - \theta_1 = \epsilon,$$
$$\theta_3 - \theta_2 = \epsilon,$$
$$\cdots\cdots\cdots\cdots$$
$$\theta_{n+1} - \theta_n = \epsilon;$$

and therefore, by addition

$$\theta_{n+1} - \theta_1 = n\epsilon.$$

When n is even, the angles θ_{n+1} and θ_1, are measured from the same mirror, and therefore $\theta_{n+1} - \theta_1$ is the angle between the initial

and final rays; therefore *the total deviation is equal to n times the inclination of the mirrors.* The deviation is the same whatever the angle of incidence, so that *any two rays are inclined at the same angle to each other after reflexion as before incidence.*

When the ray is reflected twice, once at each mirror, the deviation of the ray is twice the angle between the mirrors. This is the principle of Hadley's Sextant.

At each reflexion the value of θ is increased by ϵ. When θ becomes greater than $\frac{1}{2}\pi$ the ray will begin to come back from the angle outwards, generally by a different path; but if the angle θ be so chosen that one of the subsequent angles is equal to $\frac{1}{2}\pi$ the corresponding incidence is direct, and the ray will return by the same path. When θ becomes greater than π reflexions will cease; for the ray becomes either parallel to one of the mirrors or meets it only when produced backwards.

If the incident ray do not lie in a plane perpendicular to the line of intersection of the mirrors, the preceding investigation will apply to the projection of the path of the ray on such a plane. If, further, we remember that the inclination of the ray to this plane changes at each reflexion just as if the ray were reflected at it, the direction of the emergent ray is completely determined. After any even number of reflexions the ray makes with the principal plane the same angle as at first, and after an odd number of reflexions, an equal angle on the other side of the plane.

15. When a ray of light passes from one medium to another by refraction, the two portions of the ray before and after incidence on the new medium are called *the incident and refracted rays;* and the acute angles which they make with the normal to the surface of separation at the point of incidence, are called the *angles of incidence and refraction*, respectively.

The angles of incidence and refraction lie always in the same plane, and their sines are to one another in an invariable ratio.

This is the fundamental law of refraction; it is established by experiments which will be described later.

The constant ratio depends on the nature of the two media and

the kind of light transmitted; it is called the *refractive index* from the first medium to the second.

If a ray of light pass from a *vacuum* into a given medium, the constant ratio is the *absolute refractive index* of that medium.

If ϕ be the angle of incidence and ϕ' the angle of refraction, as a ray passes from one medium into another, the law of refraction is expressed by the equation

$$\frac{\sin\phi}{\sin\phi'} = \mu,$$

where μ is the refractive index from one medium to the other.

16. It is an experimental law, that the path of a ray of light is reversible; in other words, if a ray travel backwards through the second medium along the direction of the refracted ray, it will after refraction into the first medium retrace the path of the incident ray.

If we denote the two media by A, B and the refractive index from A into B by μ_{ab}, and the refractive index from B into A by μ_{ba}, this experiment shows that

$$\frac{\sin\phi}{\sin\phi'} = \mu_{ab}, \quad \frac{\sin\phi'}{\sin\phi} = \mu_{ba},$$

with the previous notation; and therefore, eliminating ϕ and ϕ',

$$\mu_{ab} \cdot \mu_{ba} = 1.$$

17. Also, it is found by experiment that if a ray of light pass through any number of media bounded by parallel planes, into a medium of the same nature as that in which it was originally travelling, the initial and final directions of the ray are parallel to each other.

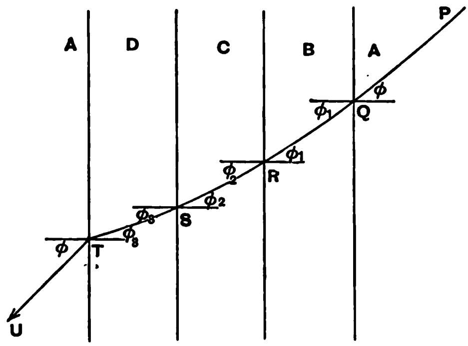
Let A be the original medium, B, C......the other media. Let ϕ be the angle of incidence on B, ϕ_1 the corresponding angle of refraction. Then ϕ_1 will be the angle of incidence on C, and so on. The final angle of refraction into A is shown to be ϕ by the experiment. Using the same notation as before to express the law of refraction at the successive surfaces, we arrive at the relations

$$\frac{\sin\phi}{\sin\phi_1} = \mu_{ab},$$

$$\frac{\sin\phi_1}{\sin\phi_2} = \mu_{bc},$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\frac{\sin\phi_n}{\sin\phi} = \mu_{ka}, \text{ say.}$$



By multiplication,

$$\mu_{ab} \cdot \mu_{bc} \cdot \mu_{cd} \cdots\cdots\cdots \mu_{ka} = 1.$$

If there are only three media, this relation becomes

$$\mu_{ab} \cdot \mu_{bc} \cdot \mu_{ca} = 1,$$

or

$$\mu_{ac} = \mu_{ab} \cdot \mu_{bc}.$$

For example, let us take the three media, air, glass and water. The values of the refractive indices from air to glass, and air to water are, respectively, $\mu_{ag} = \frac{3}{2}$, $\mu_{aw} = \frac{4}{3}$.

The preceding formula enables us to find the refractive index from glass to water.

For

$$\mu_{gw} = \mu_{ga} \cdot \mu_{aw}$$
$$= \tfrac{2}{3} \cdot \tfrac{4}{3} = \tfrac{8}{9},$$

that is, the refractive index from glass to water is $\frac{8}{9}$.

Also, let μ, μ' be the absolute refractive indices of the media A and B. Then if we denote the vacuum by the suffix v, $\mu_{ab} = \mu_{av} \cdot \mu_{vb}$. But μ_{av} is the reciprocal of μ_{va} or the reciprocal of μ, and therefore

$$\mu_{ab} = \frac{\mu'}{\mu};$$

that is, *the relative refractive index between any two media may be*

found by dividing the absolute refractive index of the second by that of the first.

The law of refraction can now be more symmetrically expressed in terms of the absolute refractive indices of the two media, μ and μ'; using the previous notation, the relation between the angles of incidence and refraction becomes

$$\mu \sin \phi = \mu' \sin \phi'.$$

18. Suppose that μ' is greater than μ; that is, suppose B to be a more highly refracting medium than A. Then if ϕ be given, the equation to determine ϕ' is,

$$\sin \phi' = \frac{\mu}{\mu'} \sin \phi.$$

This value for $\sin \phi'$ is always less than unity whatever the value of ϕ, so that a value of ϕ' can always be found for any value of ϕ. Thus when a ray of light travelling in any medium is incident on a more highly refracting medium, the law of refraction always gives a direction for the refracted ray.

But when the ray is passing from the medium B into the medium A which is less refractive, we may suppose ϕ' given, and the equation to determine ϕ is

$$\sin \phi = \frac{\mu'}{\mu} \sin \phi'.$$

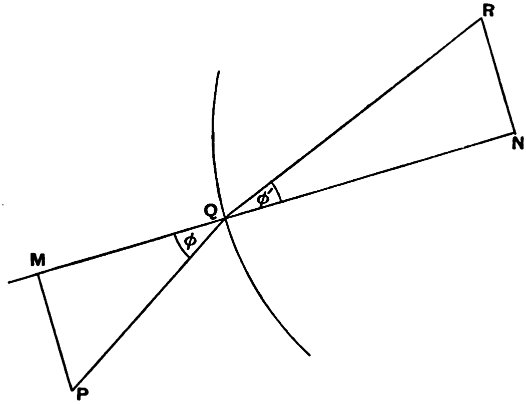
If $\sin \phi'$ is greater than μ/μ' the corresponding value for $\sin \phi$ becomes greater than unity; so that the law of refraction fails to give a real direction for the refracted ray.

The angle $\sin^{-1}(\mu/\mu')$, or, the greatest angle at which a ray of light proceeding in the more highly refractive medium can be incident on the other so as to be refracted into it, is called the *critical angle* between those media.

When a ray of light is incident on a medium less refractive than the medium in which it is moving, at an angle greater than the critical angle, the whole of the light is found to be reflected; the refracted part does not exist. This is known as *total internal reflexion.*

19. General formulæ, giving the direction cosines of the refracted ray in terms of those of the incident ray and the normal to the refracting surface, may be constructed as in the case of reflexion.

Let MQN be the normal to the refracting surface, PQR the path of the ray of light.



Measure PQ, QR along the incident and refracted rays proportional to μ and μ', the refractive indices of the media in which they are moving; and draw PM and RN perpendicular to the normal. Then since

$$\mu \sin \phi = \mu' \sin \phi',$$

the perpendicular PM is equal to the perpendicular RN.

Now the projection of PQ on any line is equal to the projection of the bent line PMQ; and the projection of QR is equal to that of the bent line QNR. But the projections of PM and NR are equal, since they are equal and parallel. Hence the difference of the projections of PQ and QR is the same as the difference of the projections of MQ and QN.

Let (p, q, r) be the direction cosines of the normal, (l, m, n) (l', m', n') the direction cosines of the incident and refracted rays respectively. Then since PQ, QR, MQ and QN are proportional to μ, μ', $\mu \cos \phi$ and $\mu' \cos \phi'$, respectively, if we take the difference of the projections on the three axes successively, we find the equations

$$\left.\begin{aligned} \mu l - \mu' l' &= (\mu \cos\phi - \mu' \cos\phi')\, p \\ \mu m - \mu' m' &= (\mu \cos\phi - \mu' \cos\phi')\, q \\ \mu n - \mu' n' &= (\mu \cos\phi - \mu' \cos\phi')\, r \end{aligned}\right\}.$$

We may substitute in these equations the values of $\cos\phi$, $\cos\phi'$ in terms of the direction cosines,

$$\cos\phi = lp + mq + nr,$$
$$\cos\phi' = l'p + m'q + n'r;$$

and then the equations are linear equations in l', m', n', l, m, n. The equations are only equivalent to two independent equations, for if we multiply them respectively by p, q, r and add them, we get an identity. The values of l', m', n' as derived from the equations are such that $l'^2 + m'^2 + n'^2 = 1$.

If we suppose that $\mu' = -\mu$, then $\phi = -\phi'$, and the refraction becomes a reflexion. Substituting these values in the general equations for the direction of the refracted ray, the equations coincide with those already given for the corresponding problem relating to reflexion. All the subsequent theorems relating to refraction will give corresponding theorems for reflexion by making the same substitution $\mu' = -\mu$.

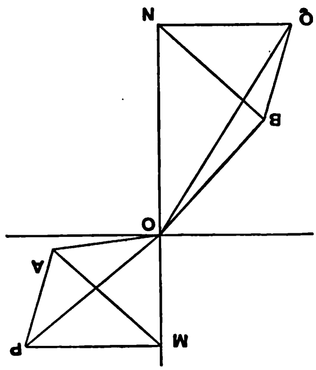
20. There are two other useful theorems relating to the incident and refracted rays which may be proved from the preceding formulæ, but the following geometrical proofs are simpler.

The angles which the incident and refracted rays make with any plane through the normal to the refracting surface, obey the law of refraction.

Also *the projections of the incident and refracted rays on any plane through the normal are connected by a law of refraction, with a refractive index depending on the inclinations of the rays to the plane.*

For let AO, OB be any two refracted rays, and let the lengths of AO, OB be taken equal to μ and μ', the refracting indices of the two media, respectively. Then if AM, BN be drawn from A and B perpendicular to the normal to the refracting surface, AM, BN will be equal and parallel

Let PO, OQ be the projections of AO, OB on any plane through the normal, P and Q being the projections of the points A, B



respectively. Then the triangles APM, BQN are equal in all respects.

Let η, η' be the acute angles which the incident and refracted rays make with the plane; ϕ, ϕ' the acute angles which the projections of these rays on the plane make with the normal. Then $AP = \mu \sin \eta$, $BQ = \mu' \sin \eta'$, and therefore, since AP is equal to BQ,

$$\mu \sin \eta = \mu' \sin \eta'.$$

This proves the first theorem.

Also $OP = \mu \cos \eta$, $OQ = \mu' \cos \eta'$; and therefore, since PM is equal to QN,

$$\mu \cos \eta \sin \phi = \mu' \cos \eta' \sin \phi',$$

which proves the second theorem.

21. *In any refraction, the greater the angle of incidence, the greater will be the angle of deviation.*

For if ϕ, ϕ' be the angles of incidence and refraction,

$$\sin \phi = \mu \sin \phi',$$

and therefore
$$\frac{\sin \phi - \sin \phi'}{\sin \phi + \sin \phi'} = \frac{\mu - 1}{\mu + 1};$$

that is,
$$\frac{\tan \frac{1}{2} (\phi - \phi')}{\tan \frac{1}{2} (\phi + \phi')} = \frac{\mu - 1}{\mu + 1},$$

or finally, $\tan \frac{1}{2}(\phi - \phi') = \frac{\mu - 1}{\mu + 1} \tan \frac{1}{2}(\phi + \phi')$.

But the deviation is equal to $\phi - \phi'$. If ϕ increase, and therefore also ϕ', $\tan \frac{1}{2}(\phi + \phi')$ will increase, since $\frac{1}{2}(\phi + \phi')$ is less than $\frac{1}{2}\pi$, and therefore the deviation will increase.

When the ray is passing into a rarer medium, we have only to suppose the ray reversed; then since the angle of refraction increases as the angle of incidence increases, the proof comes under the case just considered.

This may be seen also directly by differentiating the equation

$$\sin \phi = \mu \sin \phi'.$$

If we take logarithms and then differentiate, we see that

$$\frac{d\phi}{\tan \phi} = \frac{d\phi'}{\tan \phi'},$$

and therefore $d\phi$ is greater than $d\phi'$; so that $d(\phi - \phi')$ is positive. In other words *the deviation increases with the angles of incidence and refraction.*

But further, the preceding equation may be written in the form

$$\frac{d\phi}{d\phi'} = \frac{\mu \cos \phi'}{\cos \phi},$$

and therefore $$\left(\frac{d\phi}{d\phi'}\right)^2 = \frac{\mu^2 - \sin^2 \phi}{\cos^2 \phi},$$

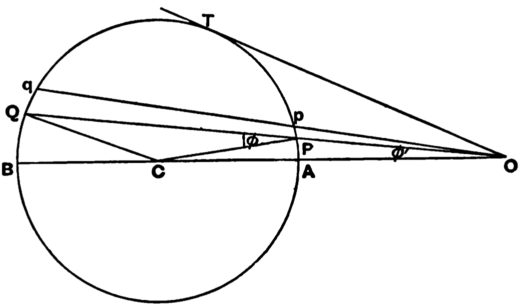
or $$\left(\frac{d\phi}{d\phi'}\right)^2 = 1 + \frac{\mu^2 - 1}{\cos^2 \phi}.$$

From this we infer that $\frac{d\phi}{d\phi'}$ increases with the angle of incidence; in other words, *as the angle of refraction increases uniformly, the deviation will increase faster and faster.*

22. We shall now give a geometrical proof of these two theorems.

Let C be the centre of any circle of radius r; take an external point O, such that $OC = \mu r$, and draw any line OPQ through O to

meet the circle in P, Q, and join CP, CQ. Then if we denote the angle CPQ by ϕ, and COP by ϕ',



$$\sin\phi : \sin\phi' = CO : CP = \mu : 1,$$

or
$$\sin\phi = \mu\sin\phi'.$$

The angles ϕ and ϕ' are therefore related like angles of incidence and refraction of a ray of light. The deviation will be the angle PCO, or D say. By varying the direction of the line OPQ from the position OAB, to the position OT in which it touches the circle, the angle ϕ will be made to increase from O to $\frac{1}{2}\pi$, and during this change D is increasing also. This proves that the deviation increases with the angle of incidence.

The angle of refraction increases from zero to the value COT; this angle represents the critical angle.

But further, as ϕ or ϕ' increases uniformly, the deviation increases faster and faster.

For let Opq be another chord of the circle close to OPQ. Then the change in D is the angle subtended at the centre by the arc Pp. And since the angle PCQ is $\pi - 2\phi$, the increase in ϕ is represented by the arc $\frac{1}{2}(Qq + Pp)$; and therefore, by subtraction, the increase in ϕ' is represented by an arc $\frac{1}{2}(Qq - Pp)$.

If we suppose ϕ, ϕ' and D to have become $\phi + x$, $\phi' + x'$ and $D + \delta$ respectively, we have

$$\frac{x}{\delta} = \frac{1}{2}\left\{\frac{Qq}{Pp} + 1\right\}$$

$$= \frac{1}{2}\left\{\frac{Oq}{OP} + 1\right\}, \text{ by similar triangles.}$$

Hence $$\frac{x}{\delta} = \tfrac{1}{2}\left\{\frac{OQ}{OP} + 1\right\} \text{ ultimately,}$$

and similarly $$\frac{x'}{\delta} = \tfrac{1}{2}\left\{\frac{OQ}{OP} - 1\right\}.$$

But as P moves from A to T, OQ becomes more and more nearly equal to OP; so that x/δ and x'/δ become smaller and smaller, which proves the proposition.

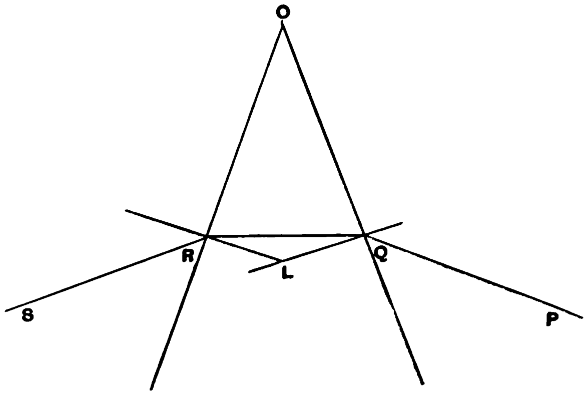
Ex. Show that $\dfrac{\sin(\phi - \phi')}{\sin(\phi + \phi')}$ increases as ϕ increases.

23. Any medium bounded by two plane faces meeting in an edge, is called a *prism*. The inclination of the faces to each other is called the *refracting angle* of the prism.

At present we shall only consider the path of rays of light which pass through the prism in a plane perpendicular to both its faces, and therefore perpendicular to the edge of the prism; we shall call such a plane a principal section of the prism.

When a ray of light passes through a prism which is more highly refractive than the surrounding medium, the deviation is, in all cases, from the refracting angle towards the thicker part of the prism.

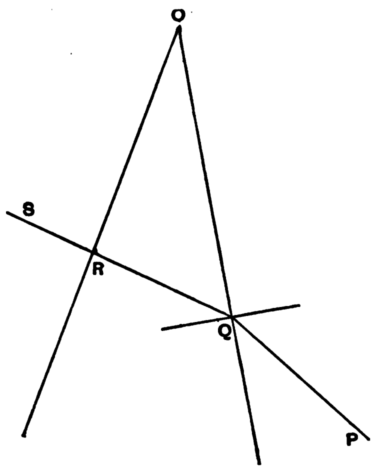
Let $PQRS$ be the course of a ray of light through a prism in a principal section QOR. Draw the normals at Q and R meeting in L. There are three cases to be considered, according as the triangle OQR is acute angled, or contains a right angle or an obtuse angle.



In the first case the rays PQ and RS lie on the sides of the

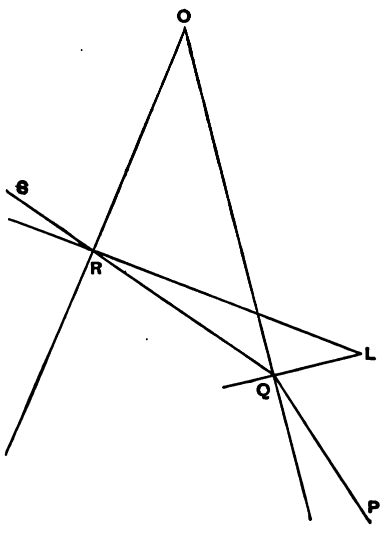
normals away from the vertex, and therefore the deviations both at ingress and egress will be away from the edge of the prism.

In the second case let one of the angles of the triangle OQR be a right angle; at that point of incidence there will be no deviation



and at the other point of incidence the deviation is away from the vertex.

In the last case, one of the angles, ORQ, is obtuse, the other



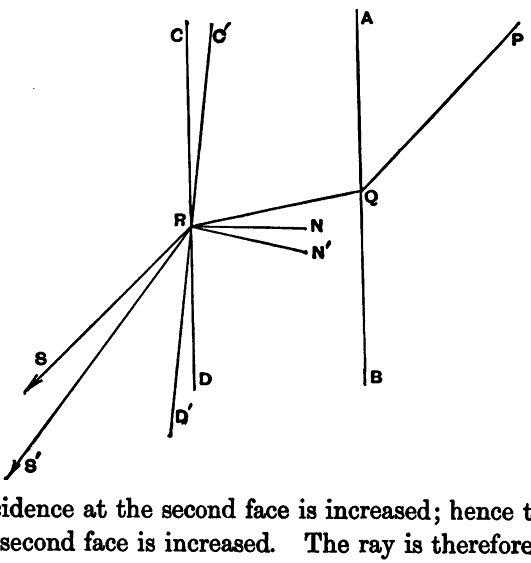
angle, OQR, being acute. Then the ray SR lies on the side of the normal towards the vertex, so that the corresponding deviation is towards the vertex, while at Q the deviation is away from the vertex. But the angle of refraction at Q is greater than that at R, the former being the exterior angle of the triangle QRL and the latter an interior angle. Hence the deviation at Q is greater than that at R, so that on the whole the deviation is away from the vertex.

If the prism be less highly refractive than the surrounding medium, all these effects are reversed.

24. This theorem may also be proved by comparing the action of a prism with that of a plate.

When a ray of light passes through a plate bounded by two parallel faces, it emerges parallel to its original direction. Let $PQRS$ be the path of a ray through such a plate bounded by the faces AB, CD. Let RN be the normal at the second face.

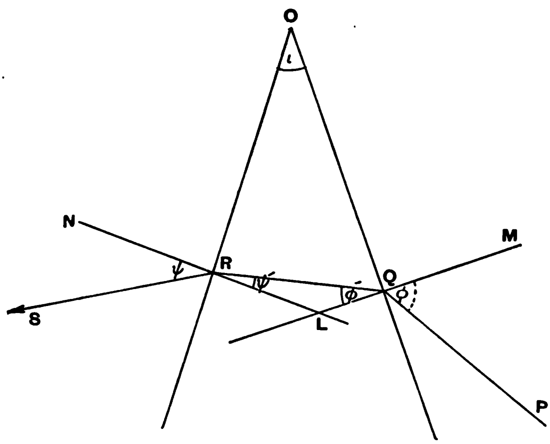
Now suppose the second face turned about R towards AB, in such a way as to make a prism whose edge is perpendicular to the plane of the ray. Let RN' be the new position of the normal to the second face and RS' the emergent ray. Then in the figure, the



angle of incidence at the second face is increased; hence the deviation at the second face is increased. The ray is therefore deviated towards the thicker part of the prism.

Similarly, if the second face *CD* be turned in the opposite direction, the deviation at the second face will be diminished and the same result will follow.

25. Let *PQRS* be the path of a ray through a prism whose



edge is at *O*, and whose refracting angle is ι.

Draw the normals at *Q* and *R*, *LQM* and *LRN* respectively, meeting in *L*.

Let ϕ, ϕ' be the angles of incidence and refraction at *Q*, and let ψ, ψ' be the angles of emergence and incidence at *R*, respectively. We shall consider ϕ and ψ as positive when they are measured from the normal towards the thicker part of the prism, so that ϕ' and ψ' will be positive when they are measured from the normals towards the vertex. In the figure ϕ, ϕ', ψ, ψ' are all positive.

By the law of refraction we have

$$\left.\begin{array}{l}\sin\phi = \mu\sin\phi' \\ \sin\psi = \mu\sin\psi'\end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Also, the angles at the base of the triangle *OQR* are respectively $\frac{1}{2}\pi - \phi'$, and $\frac{1}{2}\pi - \psi'$, hence

$$\iota + \tfrac{1}{2}\pi - \phi' + \tfrac{1}{2}\pi - \psi' = \pi,$$

or

$$\phi' + \psi' = \iota \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

This result is also true when the triangle *OQR* is obtuse; in this case one of the angles ϕ' or ψ' would be negative.

If γ be the critical angle for the medium, then ϕ' and ψ' can never be greater than γ. If, therefore, the refracting angle of the prism be greater than 2γ, no ray can pass through the prism. If ι be greater than γ, ϕ' and ψ' must always both be positive.

Let D be the whole deviation of the ray as it passes through the prism. Then at the first refraction the ray is deviated through an angle $\phi - \phi'$, and at the second refraction it is further deviated through an angle $\psi - \psi'$. Therefore

$$D = \phi - \phi' + \psi - \psi',$$

or

$$D = \phi + \psi - \iota \quad\quad (3).$$

The whole theory of the path of a ray of light in a principal section through a prism is contained in the equations (1), (2) and (3).

26. *The deviation is a minimum when the ray of light passes symmetrically through the prism.*

Let ϕ_0 be the value of ϕ for this symmetrical path, and let ϕ gradually increase from ϕ_0. Then ϕ' and ψ' increase and decrease, respectively, by equal increments; hence, since ϕ' becomes greater than ψ', the deviation at the first face increases faster than that at the second face diminishes, so that on the whole the total deviation increases. The same result is easily seen to be true even after ψ' becomes negative (if it does become negative before ϕ reaches $\frac{1}{2}\pi$). Hence as ϕ increases from ϕ_0 the deviation continually increases.

If ϕ diminishes from ϕ_0, then ψ increases from ψ_0, and we have only to consider the reversed ray to see that the same result follows.

Hence, *when the ray of light passes symmetrically through the prism, the deviation is a unique minimum.*

The theorem may also be proved by means of the formulæ of the preceding article.

The equations (1) are

$$\left.\begin{array}{l}\sin\phi = \mu\sin\phi' \\ \sin\psi = \mu\sin\psi'\end{array}\right\},$$

and therefore adding,

$$\sin\phi + \sin\psi = \mu(\sin\phi' + \sin\psi'),$$

or $2\sin\frac{1}{2}(\phi+\psi)\cos\frac{1}{2}(\phi-\psi) = 2\mu.\sin\frac{1}{2}(\phi'+\psi')\cos\frac{1}{2}(\phi'-\psi')$,

that is
$$\sin\tfrac{1}{2}(D+\iota) = \mu\sin\tfrac{1}{2}\iota\,.\,\frac{\cos\frac{1}{2}(\phi'-\psi')}{\cos\frac{1}{2}(\phi-\psi)}.$$

Suppose that ϕ and ψ are unequal, say ϕ is greater than ψ. Then the deviation $\phi-\phi'$ is greater than the deviation $\psi-\psi'$; therefore $\phi-\psi$ is greater than $\phi'-\psi'$, and therefore $\cos\frac{1}{2}(\phi'-\psi')$ is greater than $\cos\frac{1}{2}(\phi-\psi)$.

Similarly $\cos\frac{1}{2}(\phi'-\psi')$ is greater than $\cos\frac{1}{2}(\phi-\psi)$ if ψ be greater than ϕ. Hence in all cases in which ϕ and ψ are unequal $\sin\frac{1}{2}(D+\iota)$ is greater than $\mu\sin\frac{1}{2}\iota$.

But when $\phi=\psi$, $\sin\frac{1}{2}(D+\iota) = \mu\sin\frac{1}{2}\iota$.

Hence when $\phi=\psi$, D is a unique minimum.

Ex. 1. Show that

$$\sin^2\tfrac{1}{2}(D+\iota) = \sin^2\tfrac{1}{2}\iota + \frac{(\mu^2-1)\sin^2\frac{1}{2}\iota}{1-\sin^2\frac{1}{2}(\phi-\psi)\sec^2\frac{1}{2}\iota},$$

and thence show that the deviation is a minimum when $\phi=\psi$.

Ex. 2. Prove that $(\mu^2-1)\sin^2\iota = 4\cos s\cos(s-\iota)\cos(s-\phi)\cos(s-\phi')$,

where
$$2s = \phi+\phi'+\iota.$$

27. When the refracting angle of the prism is small, then the deviation will be small. In this case

$$\psi' = \iota-\phi',$$
$$\psi = \iota+D-\phi.$$

Hence,
$$\sin(\iota+D-\phi) = \mu\sin(\iota-\phi'),$$

or, since ι and D are small,

$$(\iota+D)\cos\phi - \sin\phi = \mu\iota\cos\phi' - \mu\sin\phi';$$

therefore
$$D\cos\phi = \iota\{\mu\cos\phi' - \cos\phi\},$$

or
$$D = \iota\left\{\frac{\mu\cos\phi'}{\cos\phi} - 1\right\}.$$

If the ray passes nearly perpendicularly through the prism ϕ and ϕ' will both be small, so that to the third order of small quan-

tities, the value of the deviation becomes

$$D = (\mu - 1)\,\iota,$$

which, to this approximation, is *independent of the angle of incidence.*

28. We shall next suppose that the ray does not lie in a principal plane of the prism.

Let the same notation as before be applied to the projections of the path of the light on a principal plane. Also let η, η' be the inclinations of the incident and refracted rays to the principal plane at the first refraction, ξ, ξ' the inclinations of the refracted and incident rays to the same plane, at the second refraction, respectively. Then by § 20

$$\left.\begin{aligned}\sin\eta &= \mu\sin\eta'\\ \sin\xi &= \mu\sin\xi'\end{aligned}\right\}.$$

Also, ξ' and η' denote the inclination of the same ray to the same plane, and therefore $\xi' = \eta'$ and $\xi = \eta$.

This proves that *the incident and emergent rays are equally inclined to the principal plane, or to the refracting edge of the prism.*

Further, there are the equations of refraction

$$\left.\begin{aligned}\sin\phi\cos\eta &= \mu\sin\phi'\cos\eta'\\ \sin\psi\cos\eta &= \mu\sin\psi'\cos\eta'\end{aligned}\right\},$$

and $$\phi' + \psi' = \iota.$$

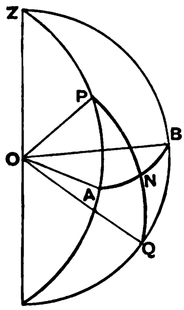
These equations contain the whole theory of the refraction of a ray through a prism.

29. We now proceed to find the deviation produced by the prism. If D_0 be the deviation of the projection of the rays on a principal plane, we shall have

$$D_0 = \phi + \psi - \iota.$$

Let OAB be the principal plane, OA, OB the projections of the incident and emergent rays on this plane, OP, OQ these rays themselves; and let all these lines be terminated on a sphere whose centre is O. Then the arc AB represents D_0, and the arc

PQ represents the complete deviation. Also the arcs AP and BQ are each equal to η, so that PQ will bisect AB in N. Then from



the right-angled triangle PAN, we deduce the equation

$$\cos \tfrac{1}{2} D = \cos \tfrac{1}{2} D_0 \cos \eta,$$

which determines the total deviation.

From this equation we see that D is always greater than D_0. Now the minimum value of D_0 is found by the same method as before; the deviation will be a minimum as the prism is turned about its refracting edge, when $\phi = \psi$; and we deduce as before the equation

$$\sin \tfrac{1}{2} (D_0 + \iota) \cos \eta = \mu \sin \tfrac{1}{2} \iota \cos \eta'.$$

In this equation if we put $\eta = 0$, we get the same value for the minimum deviation as before; this value is less than D_0, for $\cos \eta'$ is greater than $\cos \eta$. It follows, therefore, that *the deviation of a ray by a prism is least when the ray passes through the prism in a principal plane and when the angles of incidence and emergence are equal.*

EXAMPLES.

1. If $T=0$, $N=0$ be the equations to the tangent and normal at any point P of a reflecting curve, ξ, η the coordinates of the radiant point, show that the equation to the ray reflected from the curve at P is

$$\frac{T}{T'} + \frac{N}{N'} = 0,$$

where T', N' are the values of T, N when ξ, η are substituted for the current coordinates.

2. Without knowing the angles of a triangular prism, show that its refractive index can be determined by observing the minimum deviations of rays passing through the prism in the neighbourhood of the three angles; and if these deviations be denoted by 2α, 2β, 2γ, then μ is given by

$$\mu^3 - \mu^2 (\cos\alpha + \cos\beta + \cos\gamma) + \mu\{\cos(\beta+\gamma) + \cos(\gamma+\alpha) + \cos(\alpha+\beta)\} - \cos(\alpha+\beta+\gamma) = 0.$$

3. Two prisms of refractive indices μ, ν and angles α, β are placed with faces in contact, edges parallel and angles in opposite directions. Find equations to determine the angle of incidence when there is no deviation; and in this case, writing $2\mu^2 - 1 = m$ and $2\nu^2 - 1 = n$, show that if

$$m \sin^2\alpha + n \sin^2\beta = \sin^2(\alpha - \beta)$$

the angles of incidence and emergence are $x + \alpha$, $x + \beta$ respectively, where

$$m \cos 2(x+\beta) + n \cos 2(x+\alpha) = \sin^2(\alpha - \beta) - mn.$$

4. The refractive indices of three rays with respect to a given prism are μ_1, μ_2, μ_3; show that if D_1, D_2, D_3 their minimum deviations through it are in Arithmetical Progression, then

$$\frac{\sin \frac{1}{2}D_2}{\mu_2} = \frac{\sin \frac{1}{2}D_1 + \sin \frac{1}{2}D_3}{\mu_1 + \mu_3}.$$

5. Two prisms of the same vertical angle but of different refractive indices are placed in contact with their edges parallel and their angles turned opposite ways; prove that the deviation due to the system of a ray which is incident perpendicularly on the first surface of the system increases with the angle of the prisms.

6. A prism, refractive index μ' and refracting angle 60^0, is enclosed between two others of refractive indices μ and angle 60^0, their edges being turned the opposite way to that of the first. Show that if a ray passes through without deviation, its course must be symmetrical, and that

$$3\mu^2 = \mu'^2 + \mu' + 1.$$

7. If n equal and uniform prisms be placed on their ends with their edges outwards, symmetrically about a point on the table, find the angle of each prism in order that a ray refracted through each of them in a principal plane may describe a regular polygon. Show that the distance of the point of incidence of such a ray on each prism from the edge of the prism, bears to the distance of each edge from the common centre the ratio of

$$\sqrt{\mu^2 - 2\mu \cos\frac{\pi}{n} + 1} : \mu + 1.$$

8. A battery of n similar prisms is so arranged that a beam of light after traversing them at minimum deviation comes out in position to traverse them again; show that the angle A of the prisms is given by

$$\mu \sin\frac{A}{2} = \sin\left(\frac{\pi}{n} + \frac{A}{2}\right).$$

Taking glass, for which $\mu = \frac{3}{2}$, determine the least number of prisms with which the result can be accomplished, and draw a sketch of the necessary arrangement.

9. A direct-vision spectroscope is composed of three prisms, two of which are exactly alike and are placed each with a face in contact with the faces of the third and their vertices turned towards its blunt end. Find equations for the angles of the prisms and their refractive indices in order that a ray refracted through the three prisms may be able to emerge parallel to its direction of incidence.

If the refractive indices of the two similar prisms and the third be $\sqrt{6}$ and $\sqrt{3}+1$, respectively, and the angle of the third prism be 120^0, show that the angle of the two like prisms is $\tan^{-1}(6+3\sqrt{3})$.

10. If θ, ϕ be the angles of incidence and emergence of two parallel rays passing through a prism in a plane perpendicular to the edge; d_1, d_2 the distances between these rays before incidence and after emergence, show that $\dfrac{d_1}{d_2} = -\dfrac{d\phi}{d\theta}$, where $d\theta$ is any small change in θ, and $d\phi$ the corresponding change in ϕ.

11. Three plane mirrors are placed so that their intersections are parallel to each other, and the section made by a plane perpendicular to their intersections is an acute-angled triangle; a ray proceeding from a certain point of this plane after one reflexion at each mirror proceeds on its original course; prove that the point must lie on the perimeter of a certain triangle.

Prove that the ray after another reflexion at each mirror will proceed on its original path, and that the whole length of its path between the first and third reflexions at any mirror is constant and equal to twice the perimeter of the triangle formed by joining the feet of the perpendiculars.

12. Any number (n) of right-angled prisms are placed with their edges turned alternately in opposite directions, and a face of each in contact with a face of the next one, and a ray passes through with a minimum deviation. If ϕ and ψ be the angles of incidence on the first prism and emergence from the last, show that

$$\sin D = \mu_n^2 - \mu_{n-1}^2 + \mu_{n-2}^2 - \ldots\ldots - \mu_1^2,$$

if n be even, and

$$D = 2\phi - \tfrac{1}{2}\pi, \quad \phi = \psi,$$

$$\sin^2\phi = \tfrac{1}{2}(\mu_n^2 - \mu_{n-1}^2 + \ldots\ldots + \mu_1^2),$$

if n be odd.

13. The section of a prism made by a principal plane is a triangle ABC. A ray falls on AB making an angle ϕ with the normal (measured *towards* the edge) and after internal reflexion at BC emerges from AC. Employing the usual notation, show that $D = \phi + \psi + A$, $\phi' - \psi' = B - C$.

If a ray incident on AB in a direction perpendicular to BC emerge from AC parallel to the incident course, show that ABC must be an isosceles triangle.

14. A ray of light is incident in a principal plane on the base of a triangular prism and emerges at the base after internal reflexion at the other two sides. Prove that the deviation D is least when the angles of incidence and emergence are equal, and that if A is the angle between the sides of the prism, and μ the refractive index,

$$\cos \tfrac{1}{2} D = -\mu \cos A.$$

15. A ray is refracted at one face of a triangular prism in the principal plane, and after being reflected at each of the other faces emerges through the first face; show that the whole deviation is greater or less than two right angles, according as the vertical angle of the prism is less or greater than a right angle. Show also that if the angle of emergence of the ray be equal to the angle of incidence, the deviation will be a minimum when the vertical angle is less, and a maximum when it is greater, than a right angle.

16. A ray enters a prism of quadrilateral section in a principal plane and after reflexion at three sides in order emerges from the one at which it entered, making the angle of emergence equal to that of incidence but on the opposite side of the normal. Show that the section of the prism by the principal plane can be inscribed in a circle.

17. Sunlight falls on a small isosceles prism standing on a horizontal table and emerges after reflexion at the base, the edge of the prism being inclined at any angle to the sun's rays. Show that the result is the same as if the sunlight had been simply reflected at the table.

18. There are two confocal reflecting ellipses; a ray proceeds from a point P of either of them in a direction passing through one of the foci and is continually reflected between the curves. If, after $2n-1$ reflexions it returns to the point P, the length of the path is equal to n times the difference of the major axes.

19. A cylindrical pencil of light is incident on a refracting prolate spheroid in a direction parallel to the axis, the excentricity of the spheroid being e, and the refractive index μ; find the positions of the rays which emerge parallel to the axis, supposing $\mu > 1/e^2$, and show that none of the emergent rays will be parallel to the axis if $\mu < 1/e^2$.

20. The interior of an elliptic ring is a perfect reflector, and an origin of light is placed in the focus S. Show that, if P_n be the point of the nth reflection of any given ray, and if A be the vertex nearer to S, then

$$\tan \tfrac{1}{2} ASP_{2n+1} = \left(\frac{1-e}{1+e}\right)^2 \tan \tfrac{1}{2} ASP_{2n-1}.$$

21. Three plane mirrors are placed with their planes at right angles to one another. If a ray be reflected by all of them successively, its direction will be parallel to its direction at incidence.

22. If a ray of light be reflected successively at three mirrors whose normals make angles N_2N_3, N_3N_1, N_1N_2 with one another, at angles of incidence a_1, a_2, a_3, and come back in the same direction, prove that

$$\frac{\cos N_2N_3}{\cos a_1}=\frac{\cos N_3N_1}{\cos a_2}=\frac{\cos N_1N_2}{\cos a_3}.$$

23. A ray of light is reflected at each of three plane mirrors which are all parallel to one line, and the final direction of the ray is parallel to its direction before first incidence. Show that the incident ray must be parallel to one of three fixed planes, and the portions after first and second reflexions parallel to the other two; and find the positions of these planes.

24. A ray is reflected at three plane mirrors successively, so as to be parallel to its original directions after the reflexions, and the three directions which it takes are mutually at right angles to each other. Prove that the mirrors are mutually inclined at angles of 60^0.

25. A ray of light is incident from the centre of an ellipsoid, the inner surface of which is polished, and whose equation is $ax^2+by^2+cz^2=1$; prove that the equations of the ray reflected from the point (x, y, z) of the surface will be

$$\frac{\xi-x}{x(2ap^2-1)}=\frac{\eta-y}{y(2bp^2-1)}=\frac{\zeta-z}{z(2cp^2-1)},$$

where $$\frac{1}{p^2}=a^2x^2+b^2y^2+c^2z^2.$$

Prove also that all rays which after reflection pass through the line $x=y=z$, were before reflection in the surface of the cone defined by the equation

$$yz(b-c)+zx(c-a)+xy(a-b)=0.$$

26. A ray of light is incident parallel to the axis of a reflecting elliptic paraboloid whose equation is

$$by^2+cz^2=2x$$

at a point (f, g, h); show that the equations to the reflected ray are

$$\frac{x-f}{b^2g^2+c^2h^2-1}=\frac{y-g}{2bg}=\frac{z-h}{2ch}.$$

Prove that each reflected ray will pass through each of two parabolas lying in the principal planes of the paraboloid.

27. A ray of light is reflected at two plane mirrors, its direction before incidence being parallel to the plane bisecting the angle between the mirrors and making an angle θ with their line of intersection; prove that the deviation is $2\sin^{-1}(\sin\theta\sin 2a)$, where $2a$ is the angle between the planes.

More generally if D_r be the deviation after r successive reflexions,

$$\cos\tfrac{1}{2}D_{2n-1}=\sin\theta\ \sin\{(2n-1)a-\phi\},$$

$$\sin\tfrac{1}{2}D_{2n}=\sin\theta\ \sin 2na,$$

where ϕ is the angle which a plane through the intersection of the mirrors parallel to the incident ray makes with the plane bisecting the mirrors.

28. A ray of light is reflected a number of times between two plane mirrors, not in a principal plane; prove that all the reflected segments of the ray are generating lines of a hyperboloid of revolution.

29. The surface of a piece of water is covered except one narrow slit in the form of a straight line; a luminous point is placed in a given position above the surface; if this point be taken as origin of coordinates and the vertical line as axis of z, and if the equations of the slit are $x=a$, $z=-c$, show that the equation to the sheet of light in the water is

$$a^2\{(x^2+y^2)(x-a)^2+x^2(z+c)^2\}=\mu^2(x-a)^2\{a^2(x^2+y^2)+c^2x^2\}.$$

30. A ray falls on a prism whose refracting angle is $\frac{1}{2}\pi$ at a point P on one face and makes an angle θ (in any plane) with the perpendicular from P on the refracting edge. If the ray can get through without internal reflexion, show that $\theta<\cos^{-1}\sqrt{\mu^2-1}$, where μ is the refractive index.

31. The sun's light is refracted through a prism the edge of which is vertical; find the position of the refracting surfaces in order that for a given altitude of the sun the deviation of the rays of a given refractive index may be a minimum.

If z be the sun's zenith distance, ι the refracting angle, x the angle of first incidence reduced to the horizon, μ the refractive index, show that the minimum deviation D is given by the equations

$$\sin\frac{D}{2}=\sin z\sin\left(x-\frac{\iota}{2}\right),$$

$$\sin x=\mu\sin\frac{\iota}{2}\left\{1+\left(1-\frac{1}{\mu^2}\right)\cot^2 z\right\}^{\frac{1}{2}}.$$

32. A ray of light falls on a prism in a plane making an angle γ with the principal plane; show that the angle between the incident ray and the principal plane is equal to that between the same plane and the emergent ray. Prove also that if $\gamma=\frac{1}{2}\pi$ the emergent ray is parallel to a generator of the cone $\sin^2\iota\,(\mu^2r^2-x^2)=y^2$, where $r^2=x^2+y^2+z^2$, and the normal to the face of emergence is the axis of z, and the principal plane the plane of yz.

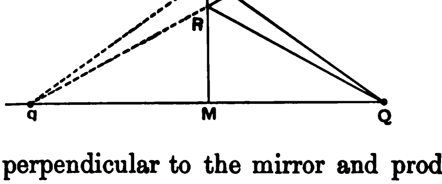
CHAPTER III.

Reflexion and Refraction of Direct Pencils.

30. Hitherto we have considered the reflexion and refraction of single rays only; we shall now consider the modifications produced in pencils of rays, by reflexion and refraction.

A pencil of rays is incident on a plane reflecting surface; to find the form of the pencil after reflexion.

Let QR be any ray diverging from a fixed point Q, and RS its course after reflexion at the mirror.

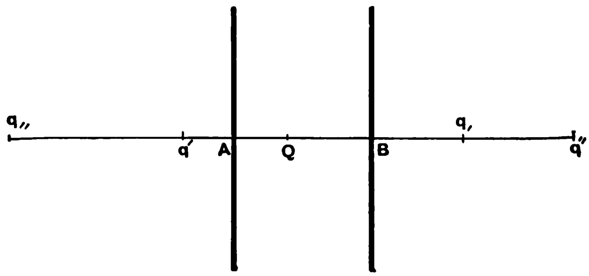


Draw QM perpendicular to the mirror and produce RS backwards to meet it in q; this can always be done, for the lines QM, QR and RS lie in one plane. Then by the law of reflexion it follows that the angle qRM is equal to the angle QRM, and therefore the triangles QRM, qRM are equal in all respects, so that $qM = QM$.

The position of q is independent of the particular ray chosen,

and the pencil after reflexion diverges from q. Thus *the foci of the incident and reflected pencils lie on the same perpendicular to the mirror, at equal distances from it, on opposite sides.*

31. *A bright point being placed between two parallel plane mirrors, required to find the foci of the reflected rays.*



Let Q be the radiant point; through Q draw a line AQB perpendicular to both surfaces and produce it indefinitely both ways.

Then, taking Aq' equal to AQ, q' will be the focus of the rays from Q after reflexion at the first mirror. These reflected rays, diverging from q' will become incident on the second mirror. If, therefore, we take Bq'' equal to Bq', q'' will be the focus of the rays after a second reflexion, and so on. Again, the rays diverging from Q and incident on the second mirror will have a focus $q_,$, where $Bq_, = BQ$; the rays diverging from this focus and incident on the first mirror will have a focus $q_{,,}$, where $Aq_{,,} = Aq_,$, and so on. Thus there are an infinite number of foci all arranged on the line AB, and becoming more and more distant after each reflexion.

The distances Qq', Qq''... may be easily calculated. For, making $QA = a$, $QB = b$, and $AB = a + b = c$, we find

$$Qq' = 2AQ = 2a,$$
$$Qq'' = BQ + Bq' = Qq' + 2BQ = 2a + 2b = 2c,$$
$$Qq''' = AQ + Aq'' = Qq'' + 2AQ = 2c + 2a,$$
$$Qq'''' = BQ + Bq''' = Qq''' + 2BQ = 2a + 2b + 2c = 4c,$$

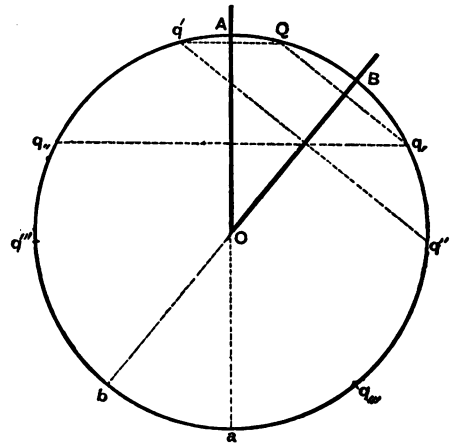
and so on.

In like manner we find

$$Qq_, = 2b, \quad Qq_{,,} = 2c, \quad Qq_{,,,} = 2c + 2b, \quad Qq_{,,,,} = 4c, \text{ \&c.}$$

32. *A bright point being placed between two plane mirrors, inclined to one another at an angle, required to find the position and number of the foci of reflected rays.*

Let OA, OB be the sections of the mirrors by a plane drawn



through the radiant point Q perpendicular to both mirrors; and let a perpendicular be drawn from Q to the mirror OA and produced to q', so that Qq' may be bisected by the mirror; then q' will be the focus of the rays after one reflexion at OA. Again, letting fall a perpendicular from q' on OB and producing it to q'', at an equal distance on the other side, q'' will be the focus of the rays after a second reflexion, and so on. In like manner we find another series of foci $q_{\prime}$, $q_{\prime\prime}$, $q_{\prime\prime\prime}$... if we first take the rays which are incident on the mirror OB.

Now, by the construction, it easily follows that $Oq' = OQ$, and in the same manner we find that $Oq'' = Oq' = OQ$. Therefore all the foci lie on the circumference of a circle whose centre is O, and radius OQ.

To determine the positions of the foci, let the arc $QA = \theta$, $QB = \theta'$ and $AB = \theta + \theta' = \iota$.

Then the arc $Qq' = \quad 2QA \quad = 2\theta$,

$$Qq'' = BQ + Bq' = Qq' + 2BQ = 2\theta + 2\theta' = 2\iota,$$

$$Qq''' = AQ + Aq'' = Qq'' + 2AQ = 2\iota + 2\theta, \text{ and so on.}$$

Similarly, $Qq_{\prime} = 2\theta'$, $Qq_{\prime\prime} = 2\iota$, $Qq_{\prime\prime\prime} = 2\iota + 2\theta'$, &c.

And, in general, the distances in the first series are

$$Qq^{(2n)} = 2n\iota, \quad Qq^{(2n+1)} = 2n\iota + 2\theta,$$

and in the second series,

$$Qq_{(2n)} = 2n\iota, \quad Qq_{(2n+1)} = 2n\iota + 2\theta'.$$

The number of images is limited; for when any one of the images falls on the arc ab, between the mirrors produced, it lies behind both mirrors, and therefore no further reflexion takes place. If the image $q^{(2n)}$ be the first to fall on the arc ab, then, since this is one of the images which lie behind the second mirror, we must have the arc $Qq^{(2n)} > QBa$; that is, $2n\iota > \pi - \theta$,

or
$$2n > \frac{\pi - \theta}{\iota}.$$

If the first image which falls on the arc ab be one of those behind the first mirror, say $Qq^{(2n+1)}$, we must have $Qq^{(2n+1)} > QAb$;

that is, $\quad 2n\iota + 2\theta > \pi - \theta', \text{ or } 2n\iota + \theta + \theta' > \pi - \theta,$

or finally,
$$2n + 1 > \frac{\pi - \theta}{\iota}.$$

This is the same result as before, $2n$ being the number of images in the first case, $2n + 1$ in the second. Therefore the whole number of images in the first series is the integer next greater than $(\pi - \theta)/\iota$; and, in like manner, the number of images in the second series may be shown to be the integer next greater than $(\pi - \theta')/\iota$.

If ι be a submultiple of two right angles, π/ι will be a whole number, and the number of images in each series will be π/ι, since θ/ι and θ'/ι are proper fractions; so that the total number of images will be $2\pi/\iota$. But in this case it happens that two of the images of the different series coincide. For if π/ι be an even integer, say $2n$, then

$$Qq^{(2n)} + Qq_{(2n)} = 2n\iota + 2n\iota = 2\pi,$$

and therefore the images $q^{(2n)}$, $q_{(2n)}$ coincide. And if π/ι be an odd integer, say $2n + 1$,

$$Qq^{(2n+1)} + Qq_{(2n+1)} = 4n\iota + 2(\theta + \theta') = (4n + 2)\iota = 2\pi,$$

and the images $q^{(2n+1)}$, $q_{(2n+1)}$ coincide.

If therefore we include the radiant point in the number, *the total number of foci is* $2\pi/\iota$.

This theory contains the principle of the *kaleidoscope*.

33. From the case of a single pencil we may now proceed to consider the way in which any object is seen by reflexion in a plane mirror.

When any object is presented to a plane mirror, every point of the object is emitting rays of light; when the rays from any point are reflected at the mirror they will proceed as if from a focus on the other side of the mirror, such that the two corresponding foci are on the same perpendicular to the mirror and at equal distances from it. To every point of the object will correspond one such focus, and the aggregate of these foci is called the *image* of the object. The image will be similar to the object and equal to it in every respect, since corresponding points of the image and object are similarly situated with respect to the mirror. But since the faces of the image and object are turned towards opposite directions, the position of the object with respect to right and left will be inverted in the image. If the eye be placed so as to receive reflected rays, they will produce the same impression as if they were radiating from a real object behind the mirror in the position occupied by the image. We may trace the rays by which the eye sees any point of the object, by drawing a pencil of lines bounded by the pupil of the eye, towards the corresponding point of the image as far as the mirror, and then joining the points of the section of the small pencil by the mirror, to the point of the object.

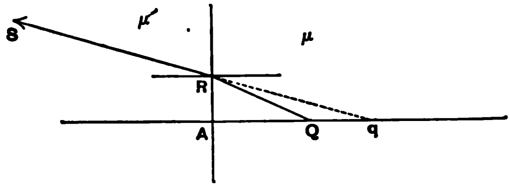
34. The pencils we shall now consider will be very slightly divergent, or in other words, the solid angles of the pencils will be very small.

When the axis of the pencil coincides with the normal to the surface on which it is incident, the incidence is said to be *direct;* in other cases the incidence is *oblique.*

In general, the rays of the pencil after refraction or reflexion do not accurately pass through a point; but there are many useful cases, where the incidence is direct, in which the rays very approximately meet in a point. We shall now consider a few of these cases.

A small pencil of light is incident directly on a plane refracting surface; to find the form of the pencil after refraction.

Let the pencil diverge from a point Q, the axis of the pencil, QA, being normal to the plane refracting surface. Let QRS be the path of any ray of light, and let RS produced backwards meet the



axis in q. Then the angle AQR is equal to the angle of incidence of the ray, and the angle AqR, equal to the angle of refraction. But if μ, μ' be the refractive indices of the two media, the law of refraction, expressed in the usual notation, is

$$\mu \sin \phi = \mu' \sin \phi'.$$

This may be written, $\dfrac{\mu}{RQ} = \dfrac{\mu'}{Rq}$.

When we consider different rays of the pencil the position of the point R will vary, and therefore the position of q will vary. But if we suppose the pencil so small, that squares and higher powers of the angles ϕ and ϕ' may be neglected, RQ is equal to AQ, and Rq to Aq to this degree of approximation; and therefore

$$\frac{AQ}{\mu} = \frac{Aq}{\mu'},$$

or as it is more usually expressed

$$\frac{u}{\mu} = \frac{u'}{\mu'},$$

where u and u' denote the lengths AQ and Aq, respectively. To this order of approximation, the point q is fixed; so that all the refracted rays produced backwards cut the axis AQ in the same point q. The point q is therefore the focus of the refracted pencil. It is sometimes called the *image* of the point Q; Q and q are also said to be *conjugate foci*.

It therefore appears *that a point and its image lie on the same normal to the surface, and on the same side of the surface; if the distance of the point from the surface changes, the distance of the image changes in the same proportion, and the point and its image move in the same direction.*

35. From the case of a single pencil we may deduce the manner and position in which an eye sees an object situated in a medium whose refractive index is different from air, as for instance an object under water. Every point of the object under water is emitting rays of light; when the rays from any point emerge in air, they will proceed from the focus conjugate to the given point. Assuming the refractive index from air to water to be $\frac{4}{3}$, the focus conjugate to a given point will lie on the same normal to the surface at $\frac{3}{4}$ of the depth. To every point of the object there will be such a corresponding focus, and the aggregate of these foci is called the image of the object. To an eye in air, the emergent rays will produce the same impression as if they proceeded from a real object occupying the position of the image. The rays by which the eye sees any point may be traced by drawing lines bounded by the pupil of the eye towards the corresponding point of the image, as far as the refracting surface, and then joining the points of the section of the small pencil by this surface, to the point of the object.

The forms of the images corresponding to different forms of object may be deduced by geometry from the preceding construction. Thus it is clear that the image of a plane, will be another plane, the two planes meeting the refracting surface in the same line at different inclinations; to a sphere, will correspond an ellipsoid of revolution whose axis of revolution is normal to the surface, and so on.

This representation of the image is only approximately true; for of the rays proceeding from any point, it is only those which are nearly normal to the surface which emerge from the image. It is therefore necessary that the object should be small and that the eye should be almost directly over it, so that all the rays may pass out in a direction nearly normal to the surface. The more accurate theory must be postponed.

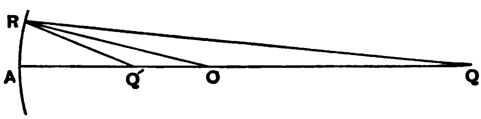
36. We shall next consider the case of a small pencil directly reflected or refracted at a spherical surface.

The most general case will not be a plane problem, but it may easily be derived from a plane problem. We shall first consider a pencil issuing from a bright point on the axis of the spherical surface; the whole system of rays will then be symmetrical about the

axis, and we need only consider rays lying in one plane through the axis, and afterwards suppose the plane system to be revolved about the axis. It will be shown that after reflexion or refraction the pencil will approximately diverge from a focus also situated on the axis. If the bright point be not on the axis of the spherical surface, we join it to the centre of the surface; the joining line is normal to the surface and therefore may be considered as itself an axis, and the problem is the same as before.

37. *Direct reflexion of a small pencil at a spherical surface.*

We first treat this as a plane problem.



Let Q be the focus of the incident pencil, QOA being the axis of the pencil, O the centre, and A the vertex of the spherical reflecting surface. Let QR be any incident ray, which after reflexion cuts the axis in Q'.

Let $AQ = x$, $AQ' = x'$, and let the radius of the circle be r.

Then by the law of reflexion, the normal RO bisects the angle between RQ and RQ', and therefore

$$RQ : RQ' = QO : OQ'.$$

Now if the pencil be small, and we neglect the squares of the small deviations of the rays from the axis, we may take

$$RQ = AQ, \quad RQ' = AQ',$$

and therefore $$x : x' = x - r : r - x',$$

that is $$x(r - x') = x'(x - r);$$

which may be written,

$$\frac{1}{x} + \frac{1}{x'} = \frac{2}{r} \quad\text{..........................}\quad (1).$$

To the degree of approximation just indicated, we may say that all rays passing through Q will after reflexion pass through Q', and *vice versa*. The points Q, Q' will be called conjugate foci; either of them may be taken to be the image of the other.

38. The formulæ we have proved connecting the distances of a pair of conjugate points from the surface include all cases that may arise. If, for instance, the reflecting surface is convex, so that AO is measured in the opposite direction, we must change the sign of r in the formula. All distances measured to the right of A are considered positive, those measured to the left, negative.

If the incident rays are parallel to the axis, so that x is infinite and positive, or infinite and negative, the corresponding value of x' in each case is

$$x' = \tfrac{1}{2}r = f, \text{ say.}$$

Hence if F be the middle point of AO, F is the focus for parallel rays proceeding in either direction. It is called the *principal focus* of the mirror.

The formula (1) may be written

$$xx' - xf - x'f = 0,$$

or

$$(x - f)(x' - f) = f^2.$$

Let u, u' be the distances of a pair of conjugate points measured from the principal focus, so that

$$\left.\begin{aligned} u &= x - f \\ u' &= x' - f \end{aligned}\right\};$$

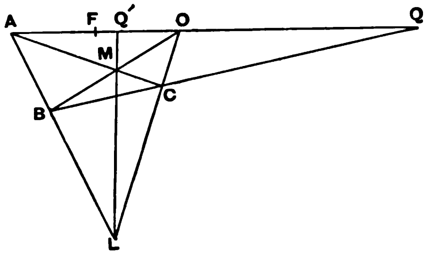
then

$$uu' = f^2.$$

From this it appears that the whole theory of reflexion at a spherical surface, whether the surface be convex towards the incident light or concave, and whether the incident pencil be convergent or divergent, may be very briefly stated by the aid of the principal focus. For let F be the principal focus, which is the point midway between the vertex of the mirror and its centre; then *a pair of conjugate foci always lie on the same side of F, and at distances, u and u' from it, such that*

$$uu' = f^2, \text{ where } f = \tfrac{1}{2}r.$$

The points Q, Q' are harmonics in regard to A, O, and therefore, by a well-known geometrical construction, the position of Q' corresponding to a given position of Q may be found by aid of the ruler only. For draw any two lines from A and O meeting in L, and from Q draw any line QCB cutting AL, OL in B, C respectively; then if AC, OB meet in M, the line LM will cut AO in the point Q' required.



The equation $\qquad (x-f)(x'-f)=f^2$

is the relation $\qquad FQ\,.\,FQ'=FO^2=FA^2,$

so that F is the centre and A, O the double points of the involution of which Q, Q' are elements.

If now the system be revolved about the axis QOA we shall have considered all the rays issuing from the point Q.

39. Now let the axis $QOQ'A$ be turned about O in all planes through a small angle, the points on it being fixed. Since the line is still normal to the surface, the points Q, Q' will still be conjugate foci. All the fixed points on the line will describe small elements of spheres whose common centre is at O. To the approximation to which we are limiting ourselves, these small elements may be taken to be plane, all these plane elements being at right angles to AO. Then, corresponding to a small plane object at Q, we shall have a plane image at Q'; the image and object will be similar and similarly situated, the lines joining corresponding points always passing through the centre of the mirror. The principal focus F will also describe a small element of a plane. This plane is called the *principal focal* plane. All pencils of parallel rays inclined at small angles to the axis of the mirror will have for foci points on the principal focal plane.

40. Let β, β' represent the linear dimensions of the object and its image, then since corresponding points lie on the same line through the centre of the sphere,

$$\frac{\beta}{\beta'}=-\frac{QO}{OQ'}$$

$$=-\frac{AQ}{AQ'}.$$

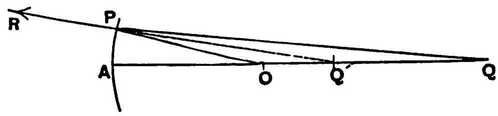
That is, $$\frac{\beta}{\beta'} = -\frac{x}{x'},$$

or $$\frac{\beta}{x} + \frac{\beta'}{x'} = 0.$$

Further details and constructions for conjugate foci will be given in the case of the direct refraction of a small pencil at a spherical surface; these will all be applicable to the present case, with the usual modifications.

41. *Direct refraction of a small pencil through a spherical surface bounding two different media.*

Let O be the centre of the spherical surface separating two media whose refractive indices are μ and μ'. Let Q be the focus



and QOA the axis of the incident pencil, and let QP be a ray in the first medium, which, proceeding from the focus Q, is refracted into the second medium along the line PR; and let PR be produced backwards to cut the axis in Q'. Draw the normal to the surface at P and let ϕ, ϕ' be the angles of incidence and refraction, respectively; then

$$\mu \sin \phi = \mu' \sin \phi'.$$

Let the angle AOP be denoted by θ; then from the triangles OPQ and OPQ', we obtain the relations

$$\frac{\sin \phi}{\sin \theta} = \frac{OQ}{QP},$$

and $$\frac{\sin \phi'}{\sin \theta} = \frac{OQ'}{Q'P};$$

and therefore $$\mu \frac{OQ}{QP} = \mu' \frac{OQ'}{Q'P}.$$

We shall suppose that the inclinations to the axis, of the rays we are considering, are small; then, if we neglect squares of θ, we can write QA for QP, and $Q'A$ for $Q'P$, so that to this approximation all rays in the plane through the axis, diverging

from Q, will after refraction pass through the point Q', determined by the relation

$$\mu \frac{OQ}{QA} = \mu' \frac{OQ'}{Q'A} \quad \text{.........................(1).}$$

The relation between the points Q and Q' is reciprocal; they are called *conjugate foci*.

Let $AQ = x$, $AQ' = x'$, and $AO = r$, x, x' and r being considered positive when they are measured from left to right. Then, in the figure, $OQ = x - r$, and $OQ' = x' - r$, and the preceding relation may be written

$$\mu \frac{(x-r)}{x} = \mu' \frac{(x'-r)}{x'},$$

and therefore,

$$\frac{\mu}{x} - \frac{\mu'}{x'} = \frac{\mu - \mu'}{r} \quad \text{.........................(2).}$$

From this equation it appears that x and x' will increase together or decrease together, so that *a pair of conjugate foci always move in the same direction.*

We might have taken the centre O for origin. Thus let $OQ = p$, $OQ' = p'$, with the same convention with regard to sign as before. Then, writing the relation (1) in the form

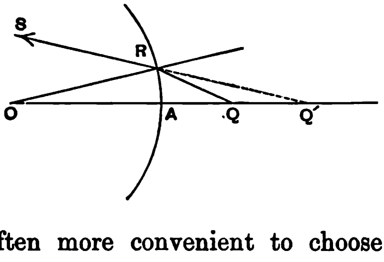
$$\mu' \frac{QA}{OQ} = \mu \frac{Q'A}{OQ'},$$

it becomes

$$\mu' \frac{p-r}{p} = \mu \frac{p'-r}{p'}$$

or

$$\frac{\mu'}{p} - \frac{\mu}{p'} = \frac{\mu' - \mu}{r}.$$



42. It is often more convenient to choose a different convention with regard to sign. Suppose light travelling from left

to right, and let $QA=x$, $Q'A=x'$, x being considered positive

when Q lies in front of A, and x' being positive when Q' lies behind A. Then the equation (2) becomes

$$\frac{\mu}{x}+\frac{\mu'}{x'}=\frac{\mu'-\mu}{r}.$$

The *principal foci* are the points conjugate to the points at infinity in the two media, and play an important part in the theory.

First, let us suppose that x' is infinite, so that the rays are parallel after refraction into the second medium; the conjugate point is then determined by the equation

$$x=\frac{\mu r}{\mu'-\mu}=f, \text{ say.}$$

Similarly if the rays are parallel to the axis in the first medium, they will, in the second medium, converge to a point Q', such that

$$x'=\frac{\mu' r}{\mu'-\mu}=f', \text{ say.}$$

These points are the *principal foci* of the surface, and f, f' its *principal focal lengths.*

The relation between x and x' may now be written in the form

$$\frac{f}{x}+\frac{f'}{x'}=1 \qquad\qquad (3).$$

The course of a ray of light is always reversible, so that this includes the case in which rays are incident on a concave spherical surface.

The focal lengths are always both positive, or both negative, so that the product ff' is always positive. They are both positive if the medium bounded by the convex surface is more highly refractive than the other, and both negative if the medium bounded by the convex surface is less refractive than the other.

43. The relation between the abscissæ of a pair of conjugate foci, when referred to the principal foci as origins, takes a very simple and convenient form.

For let A be the vertex of the spherical surface separating the two media, F, F' the principal foci of the surface, Q the focus of the rays in the first medium, Q' the focus of the rays in the second medium.

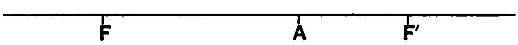
Let $QF=u$, $Q'F'=u'$, u being considered positive when Q is in front of F, and u' being considered positive when Q' is behind F'. Then the relation between x, x' may be written in the form

$$(x-f)(x-f')=ff';$$

that is, $$uu'=ff' \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

44. The relative positions of conjugate foci may be easily traced by means of the equation (4).

Since f and f' are of the same sign, it follows that u, u' will have the same sign. Two cases will have to be considered. First, suppose that the medium which is bounded by the convex surface, that is, the medium in which lies the centre of the spherical surface, is the more highly refractive, so that f and f' are both positive.

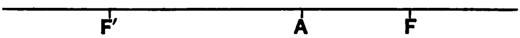


When Q is at infinity on the left, Q' will be at F'. Now Q and Q' always move in the same direction, so that as Q moves from infinity, Q' recedes beyond F', until Q reaches F, and then Q' is at infinity on the right. As Q passes F, u becomes negative and is at first very small, so that u' is negative and at first very large; thus Q' emerges from infinity on the left and follows Q as it moves along. When Q is at A, Q' coincides with it. When Q passes A, Q' follows it, but moves more slowly than Q, and when Q reaches infinity on the right, Q' coincides with F'.

So long as Q lies in the first medium, the rays in the first medium form a pencil diverging from Q; but when Q passes beyond A into the second medium, the rays in the first medium form a pencil which if produced would converge to Q, but they are intercepted by the refracting surface and never actually pass through Q. In this case, Q is called a *virtual* focus. All these

remarks apply also to the point Q'; it is a real focus only when it lies in the second medium, and a virtual focus when it lies to the left of A, in the first medium.

Next suppose that f and f' are both negative, then F lies to the right of A, in the second medium; and F' lies to the left of A, in the first medium.



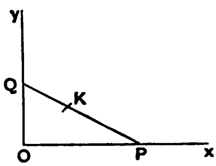
If we suppose Q to begin at infinity on the right, and to move backwards through all positions to infinity on the left, the motion of Q' may be described in exactly the same words as before. Q will be a real focus only when it lies to the left of A, and in other cases it will be a virtual focus; similarly Q' will be a real focus when it lies to the right of A, and in other cases a virtual focus.

The changes of sign and magnitude of x, x' as determined by the formula

$$\frac{f}{x}+\frac{f'}{x'}=1$$

may also be discussed by a graphical construction.

Let K be a point whose co-ordinates are (f, f'). Then x, x'



may be represented by OP, OQ, the intercepts on the axes of co-ordinates made by any line through K. If we allow a line to turn about the point K, it is easy to trace the changes in sign and magnitude of these intercepts, and therefore of x and x'.

45. A formula similar to (3), may be found for the positions of a pair of conjugate foci, if we take as origins, *any* fixed pair of conjugate foci. Let G, G' be any given pair of conjugate foci, whose distances from the principal foci are respectively g, g', so that

$$gg' = ff'.$$

Let the distances of Q, Q' from G, G' be respectively v, v',

the signs being fixed by the same convention as before. Then, with the previous notation,

$$\left.\begin{aligned} u &= v - g \\ u' &= v' - g' \end{aligned}\right\},$$

and the relation between the abscissae of conjugate foci is

$$(v-g)(v'-g') = ff' = gg';$$

which may be written in the form

$$\frac{g}{v} + \frac{g'}{v'} = 1.$$

46. We shall next consider points not on the axis of the surface.

Let the line OA turn about O through a small angle, carrying the points Q, Q', F, F', A along with it. The new line OA will still be normal to the surface, and Q, Q' will still be conjugate foci. If OA be turned through all positions making a small angle with its initial direction, the points Q, Q', F, F' describe small elements of spheres; if we neglect the squares of small quantities as before, these may be regarded as planes, cutting the axis at right angles. The planes at Q, Q' perpendicular to the axis may be called conjugate planes. Rays diverging from a point on one of the planes, after refraction at the spherical surface will converge to a point on its conjugate plane.

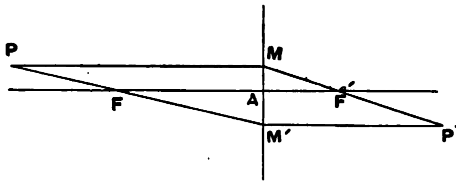
Corresponding to a small object in a plane perpendicular to the axis there will be a similar image in the conjugate plane, such that the lines joining the corresponding points of the object and its image all pass through the centre of the refracting surface.

The planes at F, F' perpendicular to the axis are called *focal planes*. Rays diverging from any point on the first focal plane will be parallel after refraction into the second medium, and conversely, any system of parallel rays in the first medium converge to a point on the second focal plane.

We have already regarded the spherical surface as approximately plane near A, and shall continue to do so. We may call it the *principal plane* of the surface.

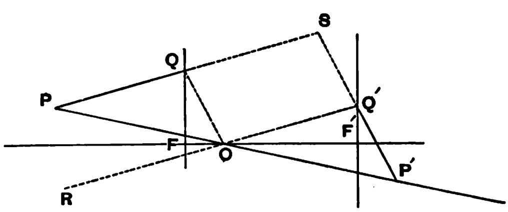
47. We can now give simple geometrical constructions for determining the focus conjugate to a given point, and for drawing the emergent ray when the incident ray is given.

Let P be a point whose conjugate focus is required. If we can trace any two rays through P they will meet in the required point. For one of the rays choose PM, parallel to the axis of the



surface, meeting the principal plane at A in M. Then MF' is the corresponding emergent ray. Also, let the ray PF meet the principal plane in M'; this ray will emerge parallel to the axis, so that if we draw $M'P'$ parallel to the axis it will meet MF' in the required point.

For either of these two rays we might have substituted the ray PO, which passes into the second medium without deviation.

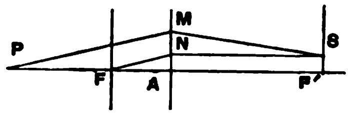


Let P be a point, PQ any ray through it meeting the first focal plane in Q; the conjugate focus of P, and the emergent ray may also be constructed in the following way.

Draw the ray PO; it will pass through into the second medium without deviation, and the conjugate focus will lie on this line. Again let ROQ' be drawn through O, parallel to the ray PQ, meeting the second focal plane in Q'. Join OQ, and from Q' draw $Q'P'$ parallel to OQ. This is the emergent ray corresponding to the incident ray PQ, and will meet PO produced in the required point. For the rays RO and PQ are initially parallel, and therefore will meet on the second focal plane; and therefore the ray PQ after refraction will pass through Q'. Also PQ and QO are two rays proceeding from Q, a point on a focal plane, and therefore they will emerge parallel to each other after refraction. But the ray

QO passes through into the second medium without deviation; therefore the emergent ray $Q'P'$ is parallel to QO.

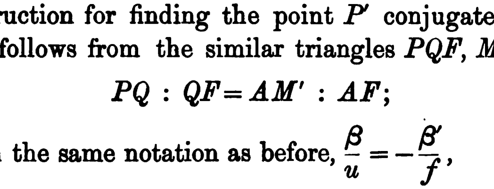
48. The emergent ray may be also constructed as follows:—



Let the incident ray meet the principal plane in M. Draw FN parallel to the incident ray to meet the principal plane in N; the ray FN will emerge along NS parallel to the axis, meeting the focal plane at F' in the point S. Then since PM and FN are parallel initially, they will converge to a point on the second focal plane, and therefore MS is the emergent ray.

These constructions will afterwards be generalised so as to give the focus conjugate to a given point and the emergent rays, after refraction through any number of spherical surfaces, with their centres arranged along an axis.

49. Let β, β' be the linear magnitudes of the object and its image, β and β' being reckoned positive or negative according as they are above or below the axis. Referring back to the geome-



trical construction for finding the point P' conjugate to a given point P, it follows from the similar triangles PQF, $M'AF$ that

$$PQ : QF = AM' : AF;$$

that is, with the same notation as before, $\frac{\beta}{u} = -\frac{\beta'}{f}$,

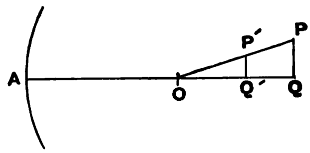
or

$$\left.\begin{aligned} \frac{\beta}{\beta'} &= -\frac{u}{f} \\ \frac{\beta'}{\beta} &= -\frac{u'}{f'} \end{aligned}\right\}.$$

and similarly,

50. Helmholtz has shown how to find an expression for the ratio of the linear magnitudes of the object and its image, in terms of the divergence of the rays before and after refraction, which is independent of the distance and focal lengths of the refracting surface.

Let PQ, $P'Q'$ be the object and image which are similar and



similarly placed, the centre of the sphere being the centre of similitude.

Thus $$\beta : \beta' = OQ : OQ'.$$

But it follows immediately from the law of refraction, as was shown in § 41, that

$$\mu \frac{OQ}{QA} = \mu' \frac{OQ'}{Q'A};$$

and therefore, if we denote QA by x, and $Q'A$ by x',

$$\frac{\mu\beta}{x} = \frac{\mu'\beta'}{x'}$$

which is a very useful formula.

But if α be the angle of divergence of any ray through Q, and α' the angle of divergence of the corresponding ray through Q',

$$\tan \alpha : \tan \alpha' = \frac{1}{x} : \frac{1}{x'};$$

and therefore,

$$\mu\beta \tan \alpha = \mu'\beta' \tan \alpha',$$

which is Helmholtz's formula.

EXAMPLES.

1. A luminous point, placed inside an equilateral triangle whose sides reflect light, is reflected in succession at the three sides taken in a definite order; the image so formed is again reflected, and so on indefinitely. Show that all the images so formed lie on one of two straight lines.

2. Two concave mirrors face each other; O, O' are their centres, and the distance AA' between the mirrors is greater than the sum of the radii. Prove that if Q, q be conjugate foci for each mirror, Qq will be the diameter of a circle which cuts orthogonally the two circles on AO, $A'O'$ as diameters.

3. If a pencil be reflected between two concave mirrors, radii ρ, σ, facing each other on the same axis at a distance a apart, show that there are two positions of the geometrical focus such that after any even number of reflexions the geometrical focus coincides with its first position, unless either both ρ and σ are greater than a, or both ρ and σ are less than a, and $\rho+\sigma > a$.

4. A pencil issuing from a point is incident upon a convex spherical refracting surface of index μ; show that the distance of the point from its conjugate focus will be a minimum, when the distance of the point from the surface is to the radius of the surface as $1 : 1+\sqrt{\mu}$.

5. A ray of light, traversing a homogeneous medium is incident upon a spherical cavity within it; supposing the limit of the magnitude of the deviation of the ray, produced by its passage through the cavity to be θ, show that the index of refraction of the medium is equal to $\sec \frac{1}{2}\theta$.

6. Rays converging to a point Q fall on a spherical surface whose centre is C; if, after one refraction, more than three rays in any plane through QC pass through the same point Q' on the axis QC, then will all the rays pass through the same point Q'.

7. Parallel rays fall on a sphere, and emerge after one internal reflexion; show that rays which are reflected at the same point of the surface are parallel after emergence; show also that, when the refractive index is greater than 2, no two rays will be reflected at the same point.

8. Find the geometrical focus after direct refraction through a hollow spherical shell bounded by two concentric spherical surfaces and filled with fluid of refractive index different from that of the shell.

9. Two spherical surfaces A, B have the same centre O; P is the geometrical focus of rays from a luminous point Q after reflexion first at the surface A and then at the surface B, and R is the geometrical focus after reflexion first at B and then at A; show that OP, OQ, OR are in harmonic progression.

10. A hemisphere of glass has its spherical surface silvered; light is incident from a luminous point Q, in the axis of figure produced, on the plane surface; show that if q is the geometrical focus of the emergent pencil, A the centre of the hemisphere, O its vertex and μ the refractive index for glass,

$$\frac{1}{Aq} - \frac{1}{AQ} = \frac{2\mu}{OA}.$$

11. A ball of glass contains a concentric spherical cavity; show that, provided the radius of the cavity do not exceed the radius of the ball divided by the index of refraction μ of the glass, it will appear to an eye at any distance from the ball to be μ times greater than it really is.

12. A sphere of a refracting substance whose index is $\sqrt{3}$ has a concentric spherical nucleus which is a reflector, whose radius is such that a ray which just enters the sphere grazes the surface of the nucleus. Prove that, if a ray, which is incident at an angle 60^0, return to the point of incidence after internal reflexions, the path within the medium will be $\frac{4}{3}$ of what it would have been if there had been no nucleus.

13. Explain why, in looking down the axis of a smooth gun barrel with an eye close to one end, a series of dark rings, images of the other end of the barrel, are seen on the surface, at distances from the eye equal to $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}$... of the length of the barrel.

14. Two equal concave mirrors of radius r are placed exactly opposite one another at a distance a, supposed greater than $2r$, apart. Rays emanating from a point on the line joining their centres are reflected alternately at the mirrors. Show that after an infinite number of reflexions the conjugate foci are distant $\frac{1}{2}\sqrt{(a^2 - 2ar)}$ from the middle point of the line joining the centres of the mirrors.

15. A transparent silver sphere is silvered at the back; prove that the distance between the images of a speck within it formed (1) by one direct refraction, (2) by one direct reflexion and one direct refraction is

$$2\mu ac\,(a-c) \div (a+c-\mu c)\,(\mu c + a - 3c),$$

where a is the radius of the sphere, and c the distance of the speck from the centre towards the silvered side.

16. Six circles are placed with their centres at the angular points of a regular hexagon. How must a ray PQ fall on one so as to fall symmetrically on all the circles in order, the index of refraction from air to one of the circles being $\sqrt{3}$?

17. A pencil diverges from a point P and passes directly through a transparent sphere whose centre is O. If Q_0 be the focus when it is not reflected inside the sphere, Q_n the focus when the pencil has been reflected $2n$ times inside the sphere, show that OQ_0, OQ_1, OQ_2...OQ_n form a series in harmonical progression, and that

$$\frac{1}{OQ_{n+1}} - \frac{1}{OQ_n} = \frac{4}{\mu r}.$$

18. A bright point is placed at the focus of a reflector which is in the form of a paraboloid of revolution; prove that the illumination, from the reflected light, of any point of a plane perpendicular to the axis of the reflector varies inversely as $(y^2 + 4a^2)^2$, where y is the distance of the point from the axis and $4a$ the latus rectum of the generating parabola.

19. A triangular prism, whose nine edges are all equal, is placed with one of its rectangular faces on a horizontal table and illuminated by a sky of uniform brightness; show that the total illumination of the inclined and vertical faces are in the ratio of $2\sqrt{3} : 1$.

20. A luminous sphere rests within a hemisphere of twice its radius, the rim of which is horizontal; find the whole illumination of the interior surface of the hemisphere; and if the sphere be raised so that its lowest point just coincides with the centre of the hemisphere, show that the illumination will be diminished in the ratio $\sqrt{5} - 1 : \sqrt{5} + 1$.

CHAPTER IV.

ELEMENTARY THEORY OF REFRACTION THROUGH LENSES.

51. A LENS is a portion of a refracting medium bounded by two surfaces of revolution which have a common axis, called the axis of the lens. In general, the surfaces of revolution are spherical or plane. If these surfaces do not meet, the lens is supposed to be bounded by a cylinder having the same axis, in addition to the surfaces of revolution.

The distance between the bounding surfaces, measured along the axis, is called the *thickness* of the lens. The thickness will generally be small in comparison with the radii of curvature of the bounding surfaces.

Lenses are classified according to their forms. A lens bounded by two convex surfaces is called a double-convex lens.

A lens bounded by two concave surfaces is called a double concave lens.

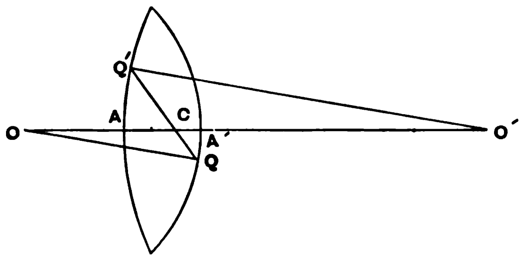
A lens of which one face is convex and the other concave is called convexo-concave or concavo-convex, according as the light first falls on the convex or concave surface, respectively.

The terms plano-convex, convexo-plane, plano-concave and concavo-plane need no further explanation.

52. We shall now consider the refraction of light through a single double-convex lens, the radii of whose faces are r, r'. The following abbreviations will be found to be convenient:

let $$\frac{r}{\mu-1}=f,\ \frac{r'}{\mu-1}=f',$$

and let the thickness of the lens be μc, μ being the refractive index of the substance of which the lens is made, when that of air is taken to be unity.



There exist two points on the axis of the lens, which are most useful in the determination of the positions of conjugate foci, and corresponding incident and emergent rays. They are a pair of conjugate foci, such that any incident ray passing through one of them, will emerge in a parallel direction through the other. These points are called the *nodal* points, and also from another property which will be pointed out later, the *principal* points of the lens.

We proceed to find the position and properties of the *nodal* points. Draw any two parallel radii OQ, $O'Q'$ of the spherical surface, and join QQ' meeting the axis in C. Then from the similar triangles OCQ, $O'CQ'$,

$$OC : O'C = r : r',$$

and therefore C is a fixed point. Any ray of light which in its path through the substance of the lens passes through C will emerge parallel to its original direction, because the tangent planes at Q, Q' are parallel to each other, and the lens will act on such a ray like a plate with parallel faces. If therefore we take N, N' the conjugate foci of C with respect to the two surfaces, a ray of light diverging from N will after the first refraction pass through C, and therefore after the second refraction will pass through N' and will emerge parallel to the original direction; in other words N, N' are the nodal points. The point C is called the *centre* of the lens.

The position of the nodal points can now be determined. The distance between the centres of the spherical surfaces is easily seen to be given by the equation

$$OO' = r + r' - \mu c,$$

and therefore, $$OC = \frac{r}{r+r'}(r + r' - \mu c)$$

$$= r - \frac{\mu cr}{r+r'}.$$

r & r' changed around from page 61

Thus $$AC = \frac{\mu cr}{r+r'} = \frac{\mu cf}{f+f'},$$

and similarly $$A'C = \frac{\mu cr'}{r+r'} = \frac{\mu cf'}{f+f'}.$$

Let h be the distance of N from A, h' the distance of N' from A', both distances being measured from the surface into the lens. Then, since N and C are conjugate foci,

$$\frac{\mu}{AC} - \frac{1}{h} = \frac{\mu - 1}{r},$$

See above.

that is, $$\frac{1}{h} = \frac{f+f'}{cf} - \frac{1}{f}.$$

From this we deduce the value of h, namely,

$$h = \frac{cf}{f+f'-c}.$$

Similarly, $$h' = \frac{cf'}{f+f'-c}.$$

53. There will be two images of a given object, formed by refraction at the two surfaces in succession, and we shall use a symmetrical notation for their positions along the axis.

Let x, x' denote the distances of the object and its first image, in front of, and behind the surface A, respectively; and let y, y' denote the distances of the final image and the first image behind and in front of the second surface, respectively. By the theory of a single refraction at a spherical surface, we get the equations

$$\left.\begin{aligned} &\frac{1}{x} + \frac{\mu}{x'} = \frac{\mu - 1}{r} \\ &\frac{1}{y} + \frac{\mu}{y'} = \frac{\mu - 1}{r'} \\ \text{and}\qquad &x' + y' = \mu c \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

If planes be drawn perpendicular to the axis of the system at the nodal points, these planes are *planes of unit magnification;*

that is, any object lying in the first plane, will have an image in the second plane, equal in all respects to the object. This theorem may also be enunciated in a slightly different manner; the line joining the points where the incident and emergent rays meet the first and second planes respectively is parallel to the axis of the system. The two planes are called the *principal planes*, and the points where they meet the axis (in this case coinciding with the nodal points), the *principal points*.

To prove this theorem, let β, $\beta_{,}$, β' denote the linear magnitudes of the object and its images, respectively. Then

$$\left.\begin{aligned}\frac{\beta}{x}+\frac{\mu\beta_{,}}{x'}=0\\ \frac{\beta'}{y}+\frac{\mu\beta_{,}}{y'}=0\end{aligned}\right\},$$

so that

$$\frac{\beta}{\beta'}=\frac{xy'}{yx'}.$$

But at the nodal points $x'/y'=r/r'$, and therefore by the equations (1), each of these ratios is equal to x/y. Hence $\beta=\beta'$.

54. If we eliminate x', y' from the equations (1), we get

$$c=\frac{1}{\frac{1}{f}-\frac{1}{x}}+\frac{1}{\frac{1}{f'}-\frac{1}{y}};$$

that is

$$c=\frac{fx}{x-f}+\frac{f'y}{y-f'}.$$

By reduction, this equation becomes

$$xy\,(f+f'-c)-fy\,(f'-c)-f'x\,(f-c)=cff'.$$

By means of this equation the positions of the *focal* points may be found; these are points such that rays diverging from them are made parallel by refraction through the lens; in other words they are the points conjugate to the points at infinity, in both directions.

If we make y indefinitely large, we get the first focal point, $x=g$, where

$$g=\frac{f(f'-c)}{f+f'-c}.$$

Similarly, the other focal point will be given by the equation $y = g'$, where

$$g' = \frac{f'(f-c)}{f+f'-c}.$$

The distance between the first focal point and the first principal point is equal to that between the second principal point and the second focal point, and this distance is called the *focal length* of the lens. If we denote this focal length by ϕ, we must have

$$\phi = g - h = g' - h',$$

which gives
$$\phi = \frac{ff'}{f+f'-c},$$

or
$$\frac{1}{\phi} = \frac{1}{f} + \frac{1}{f'} - \frac{c}{ff'}.$$

Introducing these values g, g', ϕ into the equation (2), it becomes, on dividing by $f + f' - c$,

$$xy - gy - g'x = c\phi,$$

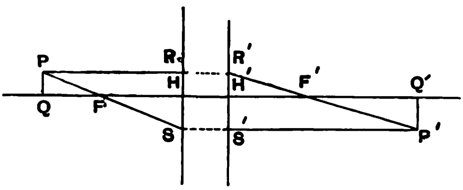
or
$$(x-g)(y-g') = gg' + c\phi$$
$$= \phi\left\{\frac{(f'-c)(f-c)}{f+f'-c} + c\right\},$$

and therefore by reduction,

$$(x-g)(y-g') = \phi^2.$$

Let the distances of a pair of conjugate points measured respectively in front of and behind the focal points, be denoted by u, v; the values of u, v are then connected by the simple formula

$$uv = \phi^2.$$



55. The position of the point P' conjugate to a given point P may now be determined by a geometrical construction. Let F, F' be the focal points, H, H' the principal points. If we can trace two rays emerging from P after refraction by the lens,

these will meet in the required point P'. For one of these rays choose the ray PR parallel to the axis, meeting the first principal plane in R; then the corresponding emergent ray will pass through R', where RR' is drawn parallel to the axis to meet the second principal plane in R'. But PR and QH are two parallel incident rays, and therefore after refraction they will meet in the focal point F'; hence $R'F'$ is the emergent ray. For the second ray choose the ray PF, meeting the principal plane in S; then the emergent ray will be parallel to the axis, through the point S', the projection of S on the second principal plane. This determines the position of P.

56. Let β, β' represent the linear magnitudes of the object PQ and its image $P'Q'$ as constructed by this process, reckoned positive if above the axis, negative if below. Then, by similar triangles,

$$PQ : QF = SH : HF.$$

But $PQ = \beta$, $QF = u$, $SH = P'Q' = -\beta'$, and $HF = \phi$; so that the relation becomes

$$\frac{\beta}{\beta'} = -\frac{u}{\phi};$$

similarly

$$\frac{\beta'}{\beta} = -\frac{v}{\phi}.$$

57. Two special cases may be noticed.

First, suppose that the thickness of the lens is very small compared with the radii of its faces; such a lens will be called a *thin* lens. In this case the points A, A' and C coincide, and the nodal points also coincide with these points. The equations then become

$$\left.\begin{aligned}\frac{1}{x} + \frac{\mu}{x'} &= \frac{\mu - 1}{r}\\ \frac{1}{y} + \frac{\mu}{y'} &= \frac{\mu - 1}{s}\end{aligned}\right\},$$

and

$$x' + y' = 0.$$

The quantities x', y' will disappear on addition, and we get

$$\frac{1}{x} + \frac{1}{y} = (\mu - 1)\left\{\frac{1}{r} + \frac{1}{s}\right\} = \frac{1}{\phi}.$$

As before, we have two focal points, each at a distance ϕ from

the lens. If the distances of a pair of conjugate points measured from these focal points be u, v, so that

$$\left.\begin{aligned} u &= x - \phi \\ v &= y - \phi \end{aligned}\right\},$$

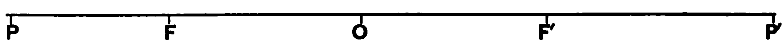
then
$$uv = \phi^2.$$

58. Next, suppose that the lens consists of a perfect sphere. In this case, we shall measure all distances from the centre of the sphere.

Let x, x' be the distances of the object and its first image, in front of, and behind, the centre, respectively, and y, y' the distances of final and first image behind, and in front of, the centre. Then we have

$$\left.\begin{aligned} &\frac{\mu}{x} + \frac{1}{x'} = \frac{\mu - 1}{r} \\ &\frac{\mu}{y} + \frac{1}{y'} = \frac{\mu - 1}{r} \\ &x' + y' = 0 \end{aligned}\right\}.$$

Hence
$$\frac{1}{x} + \frac{1}{y} = \frac{2(\mu - 1)}{\mu r} = \frac{1}{\phi}, \text{ say.}$$



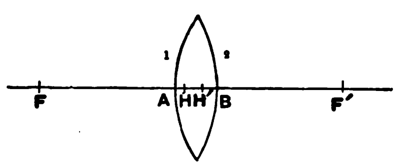
Let $OF = OF' = \phi$, so that F, F' are the focal points. Then if P, P' be a pair of conjugate points, and $PF = u$, $P'F' = v$, the same relation holds between u, v as before, namely,

$$uv = \phi^2.$$

59. We shall here trace the positions of the cardinal points for different kinds of lenses.

I. Double convex lens.

This is the typical case we have already considered; the radii r, r' are both considered positive.



We shall suppose that in each case light passes from left to right, and shall distinguish the surfaces by the figures 1, 2.

The distances of the principal points measured from the surfaces towards the substance of the lens, are respectively

$$h = \frac{cr}{r + r' - (\mu - 1) c},$$

$$h' = \frac{cr'}{r + r' - (\mu - 1) c}.$$

The distance between the principal points measured from left to right will be $\mu c - h - h'$, that is

$$HH' = \frac{(\mu - 1) c (r + r' - \mu c)}{r + r' - (\mu - 1) c} = a, \text{ say.}$$

Also the focal length is

$$\phi = \frac{rr'}{(\mu - 1) \{r + r' - (\mu - 1) c\}}.$$

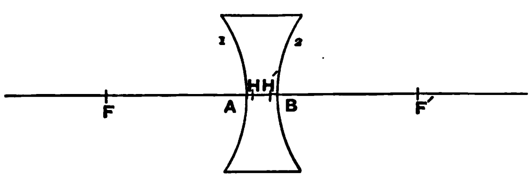
We shall suppose that the thickness of the lens is less than $r + r'$, that is $r + r'$ is $> \mu c$.

Hence h, h', a and ϕ are all positive, and the arrangement of the points is as shown in the accompanying figure.

In the limiting case in which one of the radii becomes infinite, the lens becomes a plano-convex lens. For example, suppose that r is infinite; then $h' = 0$, so that one of the principal points lies in the curved surface.

II. Double concave lens.

In this case, r and r' are both negative, so that h, h', a are all positive and ϕ negative. The arrangement of the cardinal points is shown in the figure.



If the radius of curvature of one of the faces becomes infinite, the lens is a plano-concave lens, and one of the principal points lies in the curved surface.

III. Convexo-concave lens.

We shall consider r positive and r' negative. The case in which r is negative and r' positive, may be derived from this by supposing the light to travel in the opposite direction, and the positions of the cardinal points will be the same in each case. For convenience of reference we shall write down again the values of h, h', a and ϕ, with the sign of r' changed. These are

$$h = \frac{cr}{r - r' - (\mu - 1)c},$$

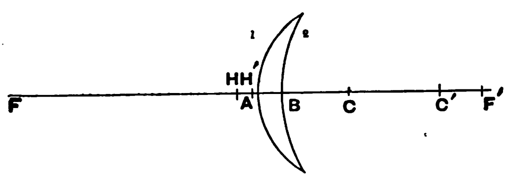
$$h' = \frac{-cr'}{r - r' - (\mu - 1)c},$$

$$a = \frac{(\mu - 1)c(r - r' - \mu c)}{r - r' - (\mu - 1)c},$$

$$\phi = \frac{-rr'}{(\mu - 1)\{r - r' - (\mu - 1)c\}}.$$

We must consider several cases separately.

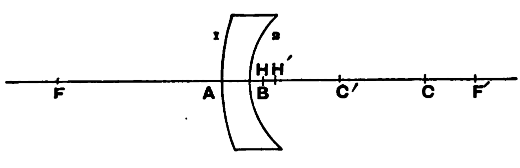
(1) Suppose that r is less than r'. Then h is negative, a is positive, and ϕ positive. The positions of the centres of curvature of the surfaces and of the cardinal points are shewn in the figure.



This lens will be thickest in the middle and thinner towards the edges.

(2) Suppose that r is greater than r', but that the centre of the surface 1 is behind the centre of the surface 2. This implies that $r > r' + \mu c$, or $r - r' > \mu c$, and *a fortiori* $r - r' > (\mu - 1)c$.

The value of h will be positive, a will be positive, but ϕ negative.



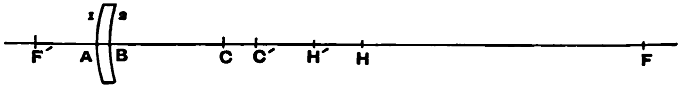
The positions of the centre of curvature and the cardinal points for this case are shown in the figure. The lens will be thinnest in the middle and will get thicker towards the edges.

(3) Suppose that r is greater than r', but that the centre of the surface 1 is in front of the centre of the surface 2. Then $r - r'$ is $< \mu c$, but may or may not be $< (\mu - 1) c$.

(α) Let $r - r'$ be $< (\mu - 1) c$. Then h is negative, a positive and ϕ positive, so that this resembles the case (1).

(β) Let $r - r'$ be $< \mu c$ but $> (\mu - 1) c$.

Then ϕ is negative, a is negative and h positive. The cardinal points are represented in the figure.



The lens in case (3), is thickest in the middle.

Summing up these results, we see that *those lenses which have positive focal length are thickest in the middle;* also *lenses thinnest in the middle have negative focal lengths.* But the converses of these statements are not true, for there is one form of lens which is thickest in the middle but yet has a negative focal length.

60. The case of a system of two lenses may also be investigated in the manner of § 52 *et seq.*

Let H, H' be the principal points, and f the principal focal length of the first lens; also let K', K be the principal points and f' the principal focal length of the second lens. Let P be any bright point, P_1 its conjugate focus after refraction by one lens, Q the conjugate focus of P_1, after refraction by the second lens. Let x, x' be the distances of P, P_1 from the principal points H, H' respectively, the distances being measured in the usual way, and also let y, y' be the distances of Q, P_1 from the principal points K, K' respectively, and let $H'K' = c$, then $x' + y' = c$.

Also
$$\left.\begin{aligned}\frac{1}{x}+\frac{1}{x'}&=\frac{1}{f}\\ \frac{1}{y}+\frac{1}{y'}&=\frac{1}{f'},\end{aligned}\right\}\dots\dots\dots\dots(1).$$

Eliminating x', y' we get the equation

$$c=\frac{1}{\frac{1}{f}-\frac{1}{x}}+\frac{1}{\frac{1}{f'}-\frac{1}{y}},$$

which becomes, after reduction,

$$xy(f+f'-c)-fy(f'-c)-f'x(f-c)=0.$$

To find the position of the focal points, we make x, y successively infinite; making y infinite, we get the position of the first focal point, $x=g$, where

$$g=\frac{f(f'-c)}{f+f'-c}.$$

Similarly, if $y=g'$ be the position of the second focal point,

$$g'=\frac{f'(f-c)}{f+f'-c}.$$

Let β, β_1, β' denote the linear magnitudes of the images at P, P_1, Q respectively, then

$$\left.\begin{aligned}\frac{\beta}{x}&=-\frac{\beta_1}{x'}\\ \frac{\beta'}{y}&=-\frac{\beta_1}{y'}\end{aligned}\right\}.$$

The principal points are points of unit magnification, and therefore to find them we must make $\beta=\beta'$; the corresponding abscissæ are such as to satisfy the equation $x/y=x'/y'$; and by equations (1), each member of this equation is equal to f/f'. Hence, reverting to the equation $x'+y'=c$, we find the values of x', y' to be

$$\left.\begin{aligned}x'&=\frac{cf}{f+f'},\\ y'&=\frac{cf'}{f+f'}\end{aligned}\right\}.$$

If h, h' be the values of x, y, corresponding to these values of x', y', we get, by equations (1),

$$\frac{1}{h}+\frac{f+f'}{cf}=\frac{1}{f},$$

or
$$h=-\frac{cf}{f+f'-c},$$

and similarly
$$h'=-\frac{cf'}{f+f'-c}.$$

These points are the principal points of the system.

If ϕ be the principal focal length of the system,

$$\phi=g-h=g'-h',$$

and therefore
$$\phi=\frac{ff'}{f+f'-c}.$$

The positions of all the cardinal points of the system have now been found, and therefore the solution is complete.

The principal points H, H' and the focal length f might just as well refer to any system of lenses instead of to one lens, and the same remark applies to the points K', K and the focal length f', and therefore this investigation shows how to combine any two given systems of lenses.

61. We shall next find the relations between the abscissæ and magnitudes of an object and its image after refraction at any number of spherical surfaces arranged symmetrically along an axis. This will include as a particular case the refraction by any number of lenses of any thicknesses arranged at intervals along the axis.

We shall suppose that there are n refracting surfaces, and that the absolute refracting indices of the several media are $\mu, \mu_1, \mu_2 \ldots\ldots \mu_n$. Let $r, r_1, r_2 \ldots\ldots r_{n-1}$ be the n radii of the surfaces, and, for brevity, suppose that

$$\frac{\mu-\mu_1}{r}=k_0, \qquad \frac{\mu_1-\mu_2}{r_1}=k_1 \ldots\ldots \qquad \frac{\mu_{n-1}-\mu_n}{r_{n-1}}=k_{n-1}.$$

Also, let the thicknesses of the media, measured along the axis be $\mu_1 t_1, \mu_2 t_2 \ldots\ldots \mu_{n-1} t_{n-1}$.

Finally, let the distance of the object from the first surface be denoted by μv, the distance of the first image also measured from the first surface by $\mu_1 v_1$, the distance of the second image measured from the second surface by $\mu_2 v_2$, and so on, and the distance of the last image measured from the last surface, by $\mu_n v_n$. We shall find the relations between these quantities, beginning at the end and reckoning backwards.

The distances of the last two images reckoned from the last surface are easily seen to be, respectively, $\mu_n v_n$, and $\mu_{n-1}(v_{n-1} - t_{n-1})$; and since these are conjugate focal distances with respect to the last surface

$$\frac{\mu_n}{\mu_n v_n} - \frac{\mu_{n-1}}{\mu_{n-1}(v_{n-1} - t_{n-1})} = \frac{\mu_n - \mu_{n-1}}{r_{n-1}},$$

or

$$\frac{1}{v_n} - \frac{1}{v_{n-1} - t_{n-1}} = -k_{n-1}.$$

This equation may be written in the form

$$v_{n-1} = t_{n-1} + \frac{1}{k_{n-1} + \frac{1}{v_n}}.$$

In exactly the same manner it may be proved that

$$v_{n-2} = t_{n-2} + \frac{1}{k_{n-2} + \frac{1}{v_{n-1}}},$$

and therefore

$$v_{n-2} = t_{n-2} + \frac{1}{k_{n-2}+}\frac{1}{t_{n-1}+}\frac{1}{k_{n-1}+}\frac{1}{v_n}.$$

Continuing this process backwards, we arrive at the equation

$$v_1 = t_1 + \frac{1}{k_1+}\frac{1}{t_2+}\frac{1}{k_2+}\dots\dots+\frac{1}{k_{n-1}+}\frac{1}{v_n}.$$

Also the distances μv, $\mu_1 v_1$, being conjugate focal distances with reference to the first surface, are connected by the relation

$$\frac{\mu}{\mu v} - \frac{\mu_1}{\mu_1 v_1} = \frac{\mu - \mu_1}{r},$$

or

$$\frac{1}{v} = k_0 + \frac{1}{v_1},$$

and therefore, finally

$$\frac{1}{v} = k_0 + \frac{1}{t_1 +} \frac{1}{k_1 +} \frac{1}{t_2 +} \frac{1}{k_2 +} \dots\dots + \frac{1}{k_{n-1} +} \frac{1}{v_n}.$$

62. Let g/h, k/l, be the last two convergents of the continued fraction

$$k_0 + \frac{1}{t_1 +} \frac{1}{k_1 +} \dots\dots + \frac{1}{k_{n-1}},$$

so that, by the properties of such fractions, $gl - hk = 1$; then the value of v will be given by the equation

$$\frac{1}{v} = \frac{v_n k + g}{v_n l + h}.$$

It will be convenient to represent the distances of the object and its final image from the first and final surfaces, respectively, by ξ, ξ'; then $\xi = \mu v$, $\xi' = \mu' v_n$, where μ' is written instead of μ_n for the refractive index of the final medium. The relation between ξ and ξ' is

$$\frac{\mu}{\xi} = \frac{\xi' k + \mu' g}{\xi' l + \mu' h},$$

or

$$k\xi\xi' + \mu' g\xi - \mu l \xi' - \mu\mu' h = 0.$$

63. The focal planes of the system are the planes conjugate to the planes at infinity.

To find the first focal plane, we must make ξ' infinite, then the rays will be parallel in the final medium. The corresponding value of ξ is

$$\xi = \frac{\mu l}{k} = \gamma_1, \text{ say.}$$

Similarly, if we make ξ infinite, so that the rays are parallel in the first medium, the value of ξ' becomes

$$\xi' = -\frac{\mu' g}{k} = \gamma_2, \text{ say.}$$

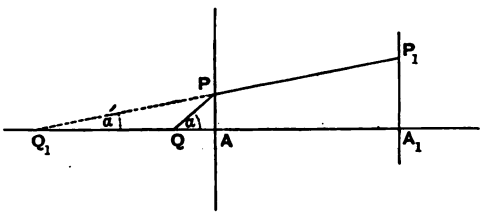
The relation between ξ, ξ' may now be written

$$\xi\xi' - \gamma_2 \xi - \gamma_1 \xi' = \frac{\mu\mu' h}{k},$$

or
$$(\xi - \gamma_1)(\xi' - \gamma_2) = \frac{\mu\mu' h}{k} - \frac{\mu\mu' lg}{k^2}$$
$$= -\frac{\mu\mu'}{k^2}\{gl - hk\},$$
that is
$$(\xi - \gamma_1)(\xi' - \gamma_2) = -\frac{\mu\mu'}{k^2}.$$

Let u, u' denote the distances of the conjugate planes from the focal planes, the same convention of sign being observed as before; then $u = \gamma_1 - \xi$, $u' = \xi' - \gamma_2$. Also let $f = -\mu/k$, $f' = -\mu'/k$. Then the relation between the abscissæ of conjugate points takes the form
$$uu' = ff'.$$

64. Let α, α_1, α_2......be the successive inclinations to the axis of a ray as it moves onwards through the different media; and let b, b_1, b_2......be the distances from the axis at which it meets the successive spherical surfaces. Also let $\mu \tan \alpha = \beta$, $\mu_1 \tan \alpha_1 = \beta_1$...



In the figure, suppose that QAA_1 represents the axis of the system, QP the incident ray, Q_1PP_1 the course of the ray after one refraction, produced backwards to meet the axis in Q_1. Then $AQ = b \cot \alpha = b\mu/\beta$. This relation may be expressed in the form $\mu/AQ = \beta/b$. In exactly the same manner it may be shewn that $\mu_1/AQ_1 = \beta_1/b$.

But $\mu/AQ - \mu_1/AQ_1 = -(\mu - \mu_1)/r$, since Q, Q_1 are conjugate foci at refraction at the spherical surface; and therefore
$$\frac{\beta}{b} - \frac{\beta_1}{b} = -k_0,$$
or
$$\beta_1 = \beta + k_0 b.$$

Also, referring back to the figure, it is easy to see that
$$b_1 = b + \mu_1 t_1 \tan \alpha_1;$$
that is
$$b_1 = b + t_1\beta_1.$$

In exactly the same manner it may be proved that

$$\left.\begin{aligned} \beta_2 &= \beta_1 + k_1 b_1 \\ b_2 &= b_2 + t_2\beta_2 \end{aligned}\right\},$$

and so on.

65. By these equations all the quantities β_1, b_1, β_2, b_2......may be expressed in terms of b and β; their values become

$$\begin{aligned} \beta_1 &= k_0\, b + \beta, \\ b_1 &= (k_0 t_1 + 1)\, b + t_1\beta, \\ \beta_2 &= \{k_1\,(k_0 t_1 + 1) + k_0\}\, b + (k_1 t_1 + 1)\,\beta, \text{ \&c.} \end{aligned}$$

The coefficients of b and β in these equations are easily seen to be, respectively, the numerators and denominators of the successive convergents to the continued fraction

$$k_0 + \frac{1}{t_1 +}\,\frac{1}{k_1 +}\,\frac{1}{t_2 +}\,\frac{1}{k_2 +}\cdots\cdots\frac{1}{+k_{n-1}}.$$

Denoting these convergents by p_1/q_1, p_2/q_2...... the equations may be written in the forms,

$$\begin{aligned} \beta_1 &= p_1 b + q_1\beta \\ b_1 &= p_2 b + q_2\beta, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ b_{n-1} &= p_{2n-2} b + q_{2n-2}\beta, \\ \beta_n &= p_{2n-1} b + q_{2n-1}\beta, \end{aligned}$$

there being n spherical surfaces.

We shall denote the last two convergents by g/h, k/l, respectively, remarking that the quantities g, h, k, l are connected by the relation $gl - hk = 1$, by the theory of continued fractions. Also, instead of the final values b_{n-1}, β_n, μ_n we shall write b', β', μ'; then the last two equations of the series become

$$\left.\begin{aligned} b' &= gb + h\beta \\ \beta' &= kb + l\beta \end{aligned}\right\}.$$

If we solve these equations, and express b, β in terms of b', β', we find by virtue of the relation $gl - hk = 1$,

$$\left.\begin{aligned} b &= lb' - h\beta' \\ \beta &= -kb' + g\beta' \end{aligned}\right\}.$$

66. We shall next find the relation between the linear dimensions of a point and its final image.

Let η, η_1, η_2... denote the linear magnitudes of the object and its successive images; then by Helmholtz' theorem

$$\mu\eta \tan\alpha = \mu_1\eta_1 \tan\alpha_1 = \mu_2\eta_2 \tan\alpha_2 \ldots;$$

that is,

$$\eta\beta = \eta_1\beta_1 = \ldots = \eta'\beta',$$

where η' denotes the linear magnitude of the final image. The value of β' has already been obtained in the form $\beta' = kb + l\beta$. Now it is easily seen that $b = -\xi \tan\alpha = -\xi\beta/\mu$, and therefore

$$\beta' = \frac{k\beta}{\mu}\left\{\frac{\mu l}{k} - \xi\right\}.$$

But $\mu l/k - \xi = \gamma_1 - \xi = u$, with the previous notation, and $f = -\mu/k$; with these abbreviations, the preceding equation becomes

$$\frac{\beta'}{\beta} = -\frac{u}{f}.$$

The relation between the linear magnitudes of the object and image is therefore

$$\frac{\eta}{\eta'} = -\frac{u}{f},$$

and from this we deduce $\dfrac{\eta'}{\eta} = -\dfrac{u'}{f'}$.

67. If we take $u = -f$ and therefore $u' = -f'$, these equations give $\eta = \eta'$; this shows that the planes $u = -f$, $u' = -f'$ are planes of unit magnification; in other words, any ray passing through the system meets these planes in two points such that the line joining them is parallel to the axis. They are called the *principal planes*, and the points where they meet the axis, the *principal points* of the system.

Let H, H' be the principal points, Q, Q' any pair of conjugate foci. Let $QH = x$, $Q'H' = x'$, the distances being measured according

to the same convention of sign as before. Then the equation $uu' = ff'$ is equivalent to $(x-f)(x'-f') = ff'$, from which we

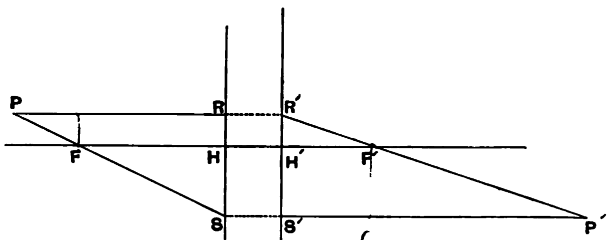
deduce the equation

$$\frac{f}{x} + \frac{f'}{x'} = 1.$$

The lengths f, f' are called the *principal focal lengths* of the system.

68. We can now give simple geometrical constructions for the focus conjugate to a given point and for an emergent ray when the incident ray is given.

Let F, F' be the principal foci, H, H' the principal points of the system.

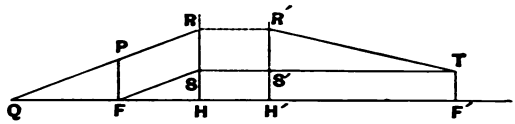


Let P be a given point, it is required to find its conjugate focus. If we can trace the course of any two rays from P, we shall be able to find P'. Take PF as one ray; let PF meet the principal plane HS in S. Draw SS' parallel to the axis to meet the other principal plane in S'; then the emergent ray will pass through S'. Also the rays FH and FS, since they diverge from a point on a focal plane, will emerge parallel to each other; if therefore we draw $S'P'$ parallel to the axis, it will be the emergent ray corresponding to PF, and will pass through the required point. For the other ray, take PR, parallel to the axis, meeting the first principal plane in R. Draw RR' parallel to the axis to meet the other principal plane in R'. Then $R'F'$ is the corresponding emergent ray; produce $R'F'$ to meet $S'P'$ in P', then P' is the point required.

69. The emergent ray may be constructed as follows:

Let QPR be the incident ray, meeting the first focal plane in P, and the first principal plane in R. Draw RR' parallel to the axis to meet the second principal plane in R'; the emergent ray will

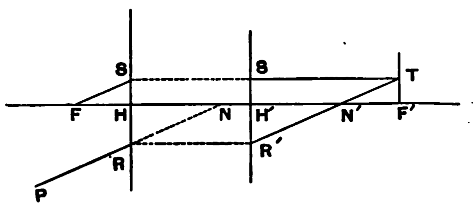
pass through R'. Again, draw a parallel incident ray from F, meeting the first principal plane in S. Draw $SS'T$ parallel to the axis meeting the second principal plane and the second focal plane



in S', T respectively; $S'T$ is the emergent ray corresponding to FS. But PR and FS are parallel, and therefore after refraction they will converge to a point on the focal plane at F'. Hence $R'T$ is the emergent ray required.

70. The best construction is effected by means of two new points, called *nodal points*. These points have their abscissæ such that $u=-f'$, $u'=-f$. Let them be denoted by N, N'; then N, N' are conjugate to each other. They also have the property *that an incident ray which passes through N will emerge from N' in a parallel direction.*

This may be shown by constructing the emergent ray corresponding to an incident ray PN passing through the point N.

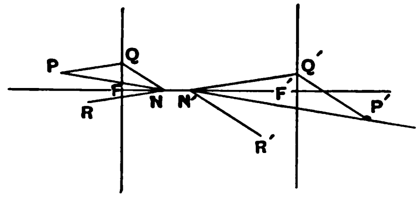


Let the points R', T be constructed as in § 69, then the emergent ray will be the line joining R' and T. But if N' be the second nodal point $F'N'=FH$ and therefore the triangles $TN'F'$, SFH are equal in all respects. Again, $H'N'=HN$, and therefore the triangles $R'N'H'$, RNH are equal in all respects. And therefore since FS, PR are parallel, the lines $N'T$, $N'R'$ are in the same straight line. This shows that the emergent ray corresponding to PN passes through N', and is parallel to the incident ray.

If the initial and final media are the same we have $f=f'$, and therefore *the nodal points coincide with the principal points.*

71. Let PQ be any incident ray through P. Let N, N' be the nodal points. Let PQ meet the first focal plane in Q. Draw $N'Q'$

parallel to PQ, meeting the second focal plane in Q'; and draw $Q'P'$ parallel to QN. Then $P'Q'$ is the emergent ray. Join PN and draw $N'P'$ parallel to it to meet the ray $Q'P'$ in P'; then P' is the



point conjugate to P. For draw RN parallel to PQ, $N'R'$ parallel to $Q'P'$. Then the rays PQ and RN are parallel, and therefore will meet on the second focal plane, after refraction. But $N'Q'$ corresponds to the ray RN, and therefore the emergent ray passes through Q'. Again PQ and QN are rays diverging from a point on the first focal plane, and therefore they will emerge parallel to each other. But QN will emerge parallel to itself; hence the emergent ray $Q'P'$ is parallel to QN. Finally, the ray PN will emerge from N' in a parallel direction, and therefore P' is conjugate to P.

72. In all cases of refraction through lenses used in air, the initial and final media are the same, and therefore $\mu=\mu'$, $f=f'=-\mu/k$, and the relation between the abscissæ of conjugate points becomes

$$uu'=f^2.$$

The nodal points also coincide with the principal points, and all the constructions depend in a simple manner on the positions of four planes and the points where they meet the axis, namely, the two focal planes and the two focal points, and the two principal planes and the two principal points.

EXAMPLES.

1. If an eye be supposed to consist of a sphere of fluid (radius r, refractive index $\frac{11}{10}$), in which is placed at a distance $\frac{2}{3}r$ from the centre a convex lens whose axis coincides with the diameter and whose focal length and refractive index in air are, respectively, $\frac{1}{3}r$ and $\frac{6}{5}$; show that the distance from the centre of the sphere for clear vision is $\frac{213}{503}r$.

2. A and B are fixed points, A being a luminous point and B the nearest point of a glass sphere with refractive index μ. C, a point on BA produced, is the image of A as seen by an eye on AB produced beyond the sphere. Show that AC is least when the radius of the sphere is $\frac{3\mu - 2}{2 - \mu} AB$.

3. Prove that the magnifying power of a thin double convex lens, the radius of each surface being ρ, when the space between the lens and an object at distance a is filled with fluid of index μ', is given by

$$\frac{1}{m} = 1 - \frac{a}{\rho} \cdot \frac{2\mu - \mu' - 1}{\mu'}.$$

4. If a ray passes through a lens without deviation and if its directions before incidence and after emergence cut the axis of the lens in two points q, q', the limiting value of qq' is a maximum or a minimum according as the thickness is equal to $\frac{(s - r)\sqrt{\mu}}{\sqrt{\mu} \mp 1}$, where r, s are the radii of the surfaces of the lens whose refractive index is μ, s being greater than r.

5. A luminous point is placed on the axis of a concave lens at a distance u from it. The light falls on a screen at a distance k behind the lens and perpendicular to the axis of the lens. If I be the illumination of the screen where it cuts the axis and if I' is what would be the illumination if the lens were removed, show that $\frac{I}{I'} = \frac{f^2 (u + k)^2}{(fu + uk + kf)^2}$.

6. If m be the linear magnifying power of a *thin* lens for an object at a distance u from the first surface, show that when the thickness t is taken into account, the magnifying power is increased by

$$m^2 \frac{\mu - 1}{s} \left(1 + \frac{\mu - 1}{r} u\right) \frac{t}{\mu}$$

in which t^2 is neglected; r and s being the radii of the two surfaces of the lens, and u the distance of the object from the first surface.

7. One side of a plate of glass is accurately plane but the other (the front) is slightly curved, forming a sphere of large radius r. Show that if a pencil is refracted through it, its focus will be displaced through a distance $\frac{\mu - 1}{\mu} \left(t + \frac{\mu u^2}{r}\right)$, where t is the thickness of the plate, and u the distance of the focus of the incident pencil from the front surface.

8. Two thin lenses of equal numerical focal length f are placed on the same axis at a distance a apart, the one nearest the origin of light being concave and the other convex; show that the least distance between an object and its final image is $a + 4f^2/a$.

9. Two thin lenses of equal focal lengths, f, are placed on the same axis at a distance a apart. P, Q are conjugate foci of any pencil refracted directly through the lenses. Show that there exist two points (O) on the axis for either of which (α and β being constants) $\frac{\alpha}{OP}+\frac{\beta}{OQ}$ is constant, provided $f<\frac{1}{2}a$ and $>-\frac{1}{2}a$; and that if $f=\frac{1}{2}a$ then $\frac{1}{OP}-\frac{1}{OQ}$ is constant.

10. A thin lens has one face silvered so as to form a mirror. If Q be the image of a point P, formed by the mirror (by two refractions and one reflexion), show that Q will be the same as if the lens were replaced by a spherical mirror whose radius R is given by the equation

$$\frac{1}{R}=\frac{\mu}{s}-\frac{\mu-1}{r},$$

r and s being the radii of the surfaces of the lens.

11. A spherical glass shell, whose outer radius is a and centre O, has its spherical cavity filled with mercury, the cavity being just large enough to prevent light from being refracted through the sphere without reflexion at the surface of the mercury. A source of light is placed at a distance c from the centre; prove that an eye, placed beyond O on the line joining the source of light to O, must, in order to receive any light, be at a distance from O whose reciprocal is not greater than

$$\frac{1}{c\mu^2}\left\{(\mu^2-2)+\frac{2}{a}(c^2-a^2)^{\frac{1}{2}}(\mu^2-1)^{\frac{1}{2}}\right\},$$

μ being the refractive index of the glass.

12. A system of $2n$ thin convex lenses of equal numerical focal length, f, are placed with their axes in the same straight line, and their centres at a distance $4f$ apart, except the two middle ones, which are at a distance $8f$ apart. Show that the focal length of a lens which must be placed midway between the two middle ones in order that the image of a bright point at a distance $4f$ in front of the first lens may be formed at an equal distance behind the last lens is $\frac{2(n+1)}{2n+1}f$.

13. A luminous point P is placed in front of a uniform sphere of glass ($\mu=\frac{3}{2}$), silvered at the back; the distance of P from the sphere and the radius of the sphere are each one foot. Prove that, if the fraction λ of the light which penetrates the sphere at or near perpendicular incidence, emerge again after reflexion at the back, the total illumination on a small screen, placed on the line from the centre of the sphere to P at a distance one foot from P, will be increased by this light beyond its value when the sphere is absent in the ratio $1+\frac{1}{81}\lambda$ to 1.

14. Show that the image of an arc of a conic whose focus is at one principal point of a thick lens, is an arc of a conic whose focus is at the other.

15. A double convex lens is formed by two equal paraboloidal surfaces cut off by planes through the focus perpendicular to the axis. Prove that for rays passing in the neighbourhood of the axis, the focal length measured from the posterior surface of the lens is $2a/(\mu^2-1)$, and the distance between a bright point and its image is a minimum when it is $2a\ (\mu+1)/(\mu-1)$, $4a$ being the latus rectum of either of the generating parabolas, and μ the refractive index of the glass.

16. The two surfaces of a lens are formed by concentric and coaxal ellipse and hyperbola, of respective eccentricities e, e', which touch one another. Show that a small pencil of light incident directly, and diverging from one focus of the ellipse, will converge to a focus of the hyperbola, if the refractive index of the lens be $\dfrac{ee'-1}{e'-e}$.

17. If m, m', m'' be the magnifying powers of a combination of any number of lenses on the same axis for objects at distances u, u', u'' from the first lens, show that

$$\frac{u'-u''}{m}+\frac{u''-u}{m'}+\frac{u-u'}{m''}=0.$$

18. If x be the distance between two objects and x' the distance between the corresponding images due to any system of lenses, and if m be the magnification of the first image and n that of the second, show that

$$\frac{x'}{x}=\frac{\mu'}{\mu}mn,$$

where μ and μ' are the refracting indices of the initial and final media.

19. If (p, p'), (q, q'), (r, r'), be three pairs of conjugate points in any lens system, prove that

$$\begin{vmatrix} pq, & pr \\ p'q', & p'r' \end{vmatrix}^2 = \frac{pq\,.\,qr\,.\,pr\,.\,p'q'\,.\,q'r'\,.\,p'r'}{ff'},$$

where pq denotes the distance between the points p, q, with similar meanings for the other quantities, and f, f' denote the focal lengths of the system.

CHAPTER V.

REFRACTION THROUGH LENSES (*continued*).

73. THE Theory of Refraction through any number of media, bounded by spherical surfaces arranged symmetrically along an axis, was first successfully developed by Gauss; we proceed to give an account of his method.

We shall take the axis of x along the axis of the system. Let the abscissæ of the vertices of the bounding spherical surfaces be $a, a_1, a_2 \ldots\ldots$, and their radii $r, r_1, r_2 \ldots\ldots$ Let $\mu, \mu_1, \mu_2 \ldots\ldots$ be the refractive indices of the media, and for brevity, let $(\mu - \mu_1)/r = k_0$, $(\mu_1 - \mu_2)/r_1 = k_1 \ldots\ldots$ and let the thicknesses of the media reckoned along the axis, be $\mu_1 t_1, \mu_2 t_2 \ldots\ldots$, so that

$$a_1 - a = \mu_1 t_1,\ a_2 - a_1 = \mu_2 t_2, \text{ \&c.}$$

Let the equations of any incident ray making a small angle with the axis of the system be

$$\left.\begin{aligned} y &= \frac{\beta}{\mu}(x-a) + b \\ z &= \frac{\gamma}{\mu}(x-a) + c \end{aligned}\right\},$$

$\mu\left(\frac{y-b}{x-a}\right) = \beta$

and after refraction at the first surface let the equations to the ray be

$$\left.\begin{aligned} y &= \frac{\beta_1}{\mu_1}(x-a) + b' \\ z &= \frac{\gamma_1}{\mu_1}(x-a) + c' \end{aligned}\right\}.$$

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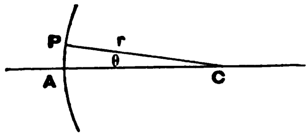
If x be the abscissæ of the point of incidence P, and θ the angle subtended at the centre of the spherical surface by the arc AP, measured from A, the vertex of the surface, then



$x = a + r(1 - \cos\theta)$, and therefore, since both the incident and refracted rays pass through this point,

$$b + \frac{\beta}{\mu} r(1 - \cos\theta) = b' + \frac{\beta_1}{\mu_1} r(1 - \cos\theta).$$

Now β, β_1 are both small, and so is θ; we shall neglect the small quantities of the third order $\theta^2\beta$, and therefore we get $b' = b$. Similarly, $c' = c$. Thus b, c are co-ordinates of the point of incidence of the ray on the first surface.

The direction cosines of the normal are $(\cos\theta, -b/r, -c/r)$, and those of the incident ray $(1, \beta/\mu, \gamma/\mu)$, and those of the refracted ray $(1, \beta_1/\mu_1, \gamma_1/\mu_1)$, nearly. Expressing the relation between these direction cosines, by § 19, we get

$$\beta - \beta_1 = -\frac{b}{r}(\mu\cos\phi - \mu_1\cos\phi_1) = -\frac{\mu - \mu_1}{r} b,$$

to our approximation. The values of β_1, γ_1 may now be written

$$\left.\begin{aligned} \beta_1 &= \beta + k_0 b \\ \gamma_1 &= \gamma + k_0 c \end{aligned}\right\},$$

and the equations of the ray after its first refraction are completely determined.

Let (b_1, c_1) be the co-ordinates of the point where the refracted ray meets the second surface, so that the equations to the refracted ray may be written in the form

$$\left.\begin{aligned} y &= \frac{\beta_1}{\mu_1}(x - a_1) + b_1 \\ z &= \frac{\gamma_1}{\mu_1}(x - a_1) + c_1 \end{aligned}\right\};$$

then if we compare these equations with the other forms of the same equations previously given, we find

$$b - \frac{\beta_1}{\mu_1} a = b_1 - \frac{\beta_1}{\mu_1} a_1,$$

and therefore
$$\left.\begin{aligned} b_1 &= b + \beta_1 t_1 \\ c_1 &= c + \gamma_1 t_1 \end{aligned}\right\}.$$

The same reasoning may be extended further; the resulting equations will have the same forms as before, namely,

$$\left.\begin{aligned} \beta_2 &= \beta_1 + k_1 b_1 \\ \gamma_2 &= \gamma_1 + k_1 c_1 \end{aligned}\right\},$$

and
$$\left.\begin{aligned} b_2 &= b_1 + \beta_2 t_2 \\ c_2 &= c_1 + \gamma_2 t_2 \end{aligned}\right\},$$

and so on.

All the successive quantities $\beta_1 \beta_2 \ldots b_1 b_2 \ldots$ may be expressed in terms of the first two quantities β and b; thus

$$\begin{aligned} &\beta_1 = k_0 b + \beta, \\ &b_1 = (k_0 t_1 + 1)\, b + t_1 \beta, \\ &\beta_2 = \{k_1 (k_0 t + 1) + k_0\}\, b + (k_1 t_1 + 1)\, \beta, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

If p_1/q_1, p_2/q_2 ...... be the successive convergents to the continued fraction

$$k_0 + \frac{1}{t_1 +}\, \frac{1}{k_1 +}\, \frac{1}{t_2 +}\, \frac{1}{k_2 +} \cdots\cdots + \frac{1}{k_{n-1}},$$

it is easy to see that the preceding equations are equivalent to the following:

$$\begin{aligned} \beta_1 &= p_1 b + q_1 \beta, \\ b_1 &= p_2 b + q_2 \beta, \\ &\ldots\ldots\ldots\ldots\ldots \\ b_{n-1} &= p_{2n-1} b + q_{2n-2} \beta, \\ \beta_n &= p_{2n-2} b + q_{2n-1} \beta. \end{aligned}$$

We shall write the final pair of equations in the form

$$\left.\begin{aligned} b' &= gb + h\beta \\ \beta' &= kb + l\beta \end{aligned}\right\}.$$

Then by the properties of convergents, $gl - hk = 1$. By exactly similar reasoning it may be shown that

$$\left.\begin{aligned} c' &= gc + h\gamma \\ \gamma' &= kc + l\gamma \end{aligned}\right\}.$$

Solving these equations, we get, by virtue of the relation $gl - hk = 1$,

$$\left.\begin{aligned} b &= lb' - h\beta' \\ \beta &= -kb' + g\beta' \end{aligned}\right\},$$

and

$$\left.\begin{aligned} c &= lc' - h\gamma' \\ \gamma &= -kc' + g\gamma' \end{aligned}\right\}.$$

The equations to the incident ray are now

$$\left.\begin{aligned} y &= \frac{\beta}{\mu}(x-a) + b \\ z &= \frac{\gamma}{\mu}(x-a) + c \end{aligned}\right\},$$

and those of the emergent ray

$$\left.\begin{aligned} y &= \frac{\beta'}{\mu'}(x-a') + b' \\ z &= \frac{\gamma'}{\mu'}(z-a') + c' \end{aligned}\right\},$$

where

$$\left.\begin{aligned} b' &= gb + h\beta \\ \beta' &= kb + l\beta \end{aligned}\right\},$$

$$\left.\begin{aligned} c' &= gc + h\gamma \\ \gamma' &= kc + l\gamma \end{aligned}\right\}.$$

In the subsequent parts of the theory, it will be sufficient to confine our attention to *one* of the equations to the incident ray; for the same reasoning will apply to the other equation in all cases. Similarly, we shall only use one equation to the emergent ray.

74. *There are two planes perpendicular to the axis which possess the property that any incident ray meets the first, and the corresponding emergent ray meets the second, in points such that the line joining them is parallel to the axis.* These planes are exceedingly useful in constructing the emergent ray corresponding to a given incident ray. We shall now prove the existence of such planes and find their positions.

The equation to the emergent ray may be written in the form

$$y = \frac{kb + l\beta}{\mu'}(x-a') + gb + h\beta,$$

or

$$y = \frac{\beta}{\mu'}\{l(x-a') + \mu'h\} + b\left\{\frac{k(x-a')}{\mu'} + g\right\}.$$

To find where the emergent ray meets the plane $x = x'$, we have only to write x' for x in this equation. Also the equation to the corresponding incident ray is

$$y = \frac{\beta}{\mu}(x - a) + b.$$

By properly choosing x and x', we may make these values of y coincide for all values of β and b, that is, for all rays. Equating the coefficients of β and b in the two equations, the necessary conditions are found to be

$$\left.\begin{aligned} \frac{l(x' - a') + \mu' h}{\mu'} &= \frac{x - a}{\mu} \\ k(x' - a') + \mu' g &= \mu' \end{aligned}\right\}.$$

From the second equation we derive at once the value of x',

$$x' = a' + \frac{\mu'}{k}(1 - g) = p_2, \text{ say.}$$

Then, from the first equation,

$$\frac{x - a}{\mu} = \frac{l}{k}(1 - g) + h = \frac{l - 1}{k},$$

and therefore

$$x = a - \frac{\mu}{k}(1 - l) = p_1, \text{ say.}$$

These two planes are called the *principal planes* and the points where they cut the axis *principal points*. They have the property that the incident ray and the emergent ray cut the two planes, respectively, in points which are the projections of each other on these planes.

75. *When the incident rays are parallel, the emergent rays will meet in a point; and for all different systems of parallel rays, the corresponding foci lie in a plane perpendicular to the axis.* Similarly, *if the emergent rays are parallel, the incident rays must proceed from some point in a fixed plane perpendicular to the axis.* These planes are called *focal planes*, and the points in which they cut the axis are called the *focal points*.

To find the position of the focal plane corresponding to incident parallel rays, we may suppose that β is fixed while b is a variable

parameter. The equation to the emergent ray being

$$y = \frac{\beta}{\mu'}\left\{l(x-a') + \mu' h\right\} + b\left\{\frac{k(x-a')}{\mu'} + g\right\},$$

and b being a variable parameter, it is clear that the ray always passes through the fixed point determined by the equations

$$\frac{k(x-a')}{\mu'} + g = 0,$$

$$y = \frac{\beta}{\mu'}\left\{l(x-a') + \mu' h\right\}.$$

The equation to the focal plane is therefore

$$x = a' - \frac{\mu' g}{k} = g_2, \text{ say.}$$

To find the equation to the other focal plane, we must make the emergent rays parallel, and therefore β' constant and b' variable. By expressing the equation to the incident ray in terms of β' and b', it may be shown in the same manner as before that the equation of the other focal plane is

$$x = a + \frac{\mu l}{k} = g_1, \text{ say.}$$

The distance of the first focal plane in front of the first principal plane is called the *first focal length* of the system; and the distance of the second focal plane behind the second principal plane is called the *second focal length*. We shall denote these focal lengths by f and f'. Their values may be deduced at once from the abscissæ of the focal planes and principal planes previously given. Thus

$$\left.\begin{aligned} p_1 - g_1 &= -\frac{\mu}{k} = f \\ g_2 - p_2 &= -\frac{\mu'}{k} = f' \end{aligned}\right\}.$$

76. There are two other points along the axes which have useful properties connected with the incident and refracted rays. They are such that if the incident ray pass through the first, the emergent ray will proceed in a parallel direction through the second. These points were discovered by Listing, who gave them the name of *nodal points*.

To find the positions of these nodal points, let us enquire in what cases the emergent ray is parallel to the incident ray. The necessary condition is

$$\frac{\beta}{\mu} = \frac{\beta'}{\mu'}.$$

This relation may be expressed in terms of b and β, and becomes

$$kb + l\beta = \frac{\mu'}{\mu}\beta,$$

or

$$b = \frac{\beta}{\mu}\left\{\frac{\mu'}{k} - \frac{\mu l}{k}\right\}.$$

If this value be substituted in the equation of the incident ray, it takes the form

$$y = \frac{\beta}{\mu}\left\{x - a + \frac{\mu'}{k} - \frac{\mu l}{k}\right\}.$$

From this equation it appears that whatever the direction of the incident ray, it passes through the point on the axis, for which

$$x = a - \frac{\mu'}{k} + \frac{\mu l}{k} = n_1, \text{ say.}$$

In the same manner it may be shown that the equation to the emergent ray takes the form

$$y = \frac{\beta'}{\mu'}\left\{x - a' - \frac{\mu}{k} + \frac{\mu g}{k}\right\};$$

and therefore the emergent ray passes through the point on the axis, for which

$$x = a' + \frac{\mu}{k} - \frac{\mu g}{k} = n_2.$$

The coordinates n_1, n_2 of the nodal points are therefore determined. From their values it is easily seen that

$$\left.\begin{aligned} n_1 - g_1 &= f' \\ g_2 - n_2 &= f \end{aligned}\right\}.$$

The nodal points are therefore within the focal points, at distances from them equal to the focal lengths f' and f, respectively.

In the very important case in which the initial and final media are the same, $\mu = \mu'$ and therefore $f = f'$; and the nodal points coincide with the principal points.

Other methods of finding the positions of these cardinal points will be given later.

77. *When a system of incident rays all proceed from one point, the emergent rays also all pass through a point.* These points are called *conjugate foci;* also, one of them is sometimes said to be the *image* of the other.

For we have seen how the incident and emergent rays depend entirely upon the values of β, b, γ, c. The conditions that the incident ray may pass through a point (ξ, η, ζ) are

$$\mu\eta = \beta(\xi - a) + \mu b,$$

and a similar equation in (ξ, ζ). Also the conditions that the emergent ray may pass through the point (ξ', η', ζ') are

$$\mu'\eta' = \beta\{l(\xi' - a') + \mu' h\} + b\{k(\xi' - a') + \mu' g\},$$

and a similar equation in ξ', ζ'. It is possible so to choose (ξ', η', ζ') as to make the second conditions merely repetitions of the first. This will necessitate the equations

$$\frac{\mu'\eta'}{\mu\eta} = \frac{\mu'\zeta'}{\mu\zeta} = \frac{l(\xi' - a') + \mu' h}{\xi - a} = \frac{k(\xi' - a') + \mu' g}{\mu}.$$

If these conditions are satisfied, the points (ξ, η, ζ), (ξ', η', ζ') are so related that any incident ray passing through the former, will after refraction pass through the other point; in other words, the two points are conjugate foci.

From the equation

$$\frac{\eta'}{\eta} = \frac{\zeta'}{\zeta},$$

we infer that *a point and its image lie in the same plane through the axis.*

78. The relation between the abscissæ of the conjugate foci is

$$k(\xi - a)(\xi' - a') + \mu' g(\xi - a) - \mu l(\xi' - a') - \mu\mu' h = 0.$$

From this equation the positions of the focal points can be deduced, and by means of these points the relation can be much simplified. To find the focal points, we make first, the incident rays, and secondly, the emergent rays, parallel; this will be done by making successively ξ and ξ' infinite. The

abscissæ of the corresponding images will be

$$\left.\begin{aligned} \xi' &= a' - \frac{\mu' g}{k} = g_2 \\ \xi &= a + \frac{\mu l}{k} = g_1 \end{aligned}\right\}.$$

The relation between ξ, ξ' may now be written

$$(\xi - a)(\xi' - a') - (g_2 - a')(\xi - a) - (g_1 - a)(\xi' - a') = \frac{\mu\mu' h}{k},$$

which may be expressed in the form

$$\{(\xi - a) - (g_1 - a)\}\{(\xi' - a') - (g_2 - a')\} = (g_1 - a)(g_2 - a') + \frac{\mu\mu' h}{k}$$

$$= \mu\mu'\left\{\frac{h}{k} - \frac{gl}{k^2}\right\} = -\frac{\mu\mu'}{k^2},$$

or
$$(\xi - g_1)(\xi' - g_2) = -ff'.$$

If we denote the distances of the conjugate foci, respectively, in front of, and behind, the principal foci, by u, u', then $u = g_1 - \xi$, $u' = \xi' - g_2$, and therefore

$$uu' = ff'.$$

79. Returning to the other coordinates of the conjugate foci, we find

$$\frac{\eta'}{\eta} = \frac{\zeta'}{\zeta} = \frac{k(\xi' - a') + \mu' g}{\mu'}$$

$$= -\frac{(\xi' - g_2)}{f'};$$

that is,

$$\frac{\eta'}{\eta} = \frac{\zeta'}{\zeta} = -\frac{u'}{f'}.$$

Inverting, and remembering the relation between u and u', this equation may be written

$$\frac{\eta}{\eta'} = \frac{\zeta}{\zeta'} = -\frac{u}{f}.$$

From these equations it follows that if $u = -f$, and therefore $u' = -f'$, then $\eta = \eta'$, $\zeta = \zeta'$. Thus, according to our previous definition, the planes $u = -f$, $u' = -f'$ are the principal planes.

If the distances of any pair of conjugate foci measured, respectively, in front of and behind the principal planes be denoted by x, x', then $u = x - f$, $u' = x' - f'$, and the relation between x, x' is

$$(x - f)(x' - f') = ff',$$

or, as it may be written,

$$\frac{f}{x}+\frac{f'}{x'}=1.$$

For the application of these various results to the geometrical construction of the corresponding incident and emergent rays, and of conjugate foci, we refer back to the previous elementary theory.

80. A single surface or a thin lens can always be found, which when placed with its vertex at the first principal point will refract the incident rays into exactly the same *directions* as the whole system of surfaces, so that if we imagine the interval between the two principal planes to be annihilated, this single refracting surface or thin lens, will give the complete emergent system. Such a surface or lens is said to be *equivalent* to the whole system of refracting surfaces. The system will be equivalent to a single refracting surface when the initial and final media are different; when the initial and final media are the same, it is equivalent to a thin lens.

To prove these propositions, we refer the equations to the rays, to the principal points as origins. The equation to the incident ray may be written

$$y=\frac{\beta}{\mu}(x-p_1)+\frac{\beta}{\mu}(p_1-a)+b$$

$$=\frac{\beta}{\mu}(x-p_1)-\frac{\beta}{k}(l-1)+b,$$

or finally,

$$y=\frac{\beta}{\mu}(x-p_1)+\frac{\beta'-\beta}{k};$$

and the corresponding emergent ray has for its equation

$$y=\frac{\beta'}{\mu'}(x-p_2)+\frac{\beta'-\beta}{k}.$$

The value of y corresponding to the point where the rays meet the principal planes is

$$y=\frac{\beta'-\beta}{k}=b_0,\ \text{say};$$

then $$\beta'=\beta+kb_0.$$

But this is the equation we should have obtained by a single refraction at a spherical surface, as may be seen by comparing it with the equations previously obtained. If μ, μ' be the refractive indices of the media, and r the radius of the surface, the value of k is

$$k = \frac{\mu - \mu'}{r},$$

and therefore

$$r = \frac{\mu - \mu'}{k}.$$

This value of r is measured from the vertex in the direction of the incident light. The system is therefore equivalent to a single refracting spherical surface whose radius is $(\mu - \mu')/k$ and whose vertex is at the first principal point.

81. When the initial and final media are the same, it is necessary to suppose a thin lens to be placed with its vertex at the principal point. The equations corresponding to the refraction through a thin lens are

$$\beta_1 = \beta + k_0 b_0,$$

$$b_1 = b_0,$$

$$\beta_2 = \beta_1 + k_1 b_1;$$

and therefore

$$\beta' = \beta + (k_0 + k_1)\, b_0.$$

This will produce the necessary refraction, provided that

$$k = k_0 + k_1.$$

If μ' be the refractive index of the substance of the lens, r, r' the radii of its two surfaces measured in the same direction as before,

$$k_0 + k_1 = \frac{\mu - \mu'}{r} + \frac{\mu' - \mu}{r'}$$

$$= (\mu - \mu')\left(\frac{1}{r} - \frac{1}{r'}\right).$$

Let ϕ be the focal length of the thin lens, defined in the usual manner, and expressed in terms of the refractive index of the initial medium, then

$$\frac{1}{\phi} = \left(\frac{\mu'}{\mu} - 1\right)\left(\frac{1}{r} - \frac{1}{r'}\right);$$

hence $$k = -\frac{\mu}{\phi},$$

or finally $$\phi = -\frac{\mu}{k}.$$

The focal length of the equivalent lens is therefore $-\mu/k$; it will be a collective lens if $-\mu/k$ be positive, and a dispersive lens if $-\mu/k$ be negative.

82. There is one case in which we cannot make use of the subsidiary points, and, as it occurs frequently in practice, it must be noticed. This is the case in which k vanishes; for then the subsidiary points are all at infinity. When $k = 0$, the equations previously given reduce to

$$gl = 1, \quad \beta' = l\beta, \quad b' = gb + h\beta.$$

If the equations to the incident ray be

$$\left.\begin{aligned} y &= \frac{\beta}{\mu}(x - a) + b \\ z &= \frac{\gamma}{\mu}(x - a) + c \end{aligned}\right\},$$

those of the corresponding emergent ray will be

$$\left.\begin{aligned} y &= \frac{l\beta}{\mu'}(x - a') + gb + h\beta \\ z &= \frac{l\gamma}{\mu'}(x - a') + gc + h\gamma \end{aligned}\right\}.$$

If we put $\alpha = a' - \frac{\mu' h}{l}$, these last may be written

$$\left.\begin{aligned} y &= \frac{l\beta}{\mu'}(x - \alpha) + gb \\ z &= \frac{l\gamma}{\mu'}(x - \alpha) + gc \end{aligned}\right\}.$$

We proceed as before to find the image of a point (ξ, η, ζ); the relations between (ξ, η, ζ) and the coördinates of the image (ξ', η', ζ') are easily seen to be

$$\frac{\mu'\eta'}{\mu\eta} = \frac{\mu'\zeta'}{\mu\zeta} = \frac{l(\xi' - \alpha)}{\xi - a} = \frac{\mu'}{\mu}g;$$

these equations become on reduction

$$\frac{\eta'}{\eta} = \frac{\zeta'}{\zeta} = g = \frac{1}{l},$$

and
$$\mu l\,(\xi' - a) = \mu' g\,(\xi - a).$$

Thus it follows that *the linear dimensions of the image are to those of the object in the constant ratio* $g : 1$ *or* $1 : l$, *wherever the object be placed.*

The case now considered will present itself with a telescope arranged for seeing very distant objects by a long-sighted person. From the formulæ, it is clear that a set of rays which are originally parallel will emerge parallel to each other. In the case of a telescope we shall have $\mu = \mu'$, and therefore by the previous equations, the tangent of the inclination of the initial ray to the axis is to that of the emergent ray in the ratio $1 : l$. Hence, according to the usual definition, l or $1/g$ is the *magnifying power* of the telescope. If l be positive, the image is erect; if l be negative, it is inverted.

Elementary Theory of Equivalent Lenses.

83. A lens is said to be *equivalent* to any number of lenses arranged at intervals along an axis when, if placed in a proper position, it will produce the same deviation in rays inclined at small angles to the axis of the system, as would be produced by the system of lenses.

We shall first suppose the incident rays to be parallel to the axis of the system, so that the position of the equivalent lens is immaterial.

The deviation produced by a thin lens may be found by supposing the lens to act like a thin prism formed by the tangent planes to the spherical surfaces at the points of incidence and emergence of the ray. The deviation will therefore be independent of the angle of incidence, for all small angles of incidence. To find the deviation, we suppose the incident ray to be parallel to the axis, and then the emergent ray will proceed to the principal focus of the lens. If y be the distance from the axis at which the ray strikes the lens, and f the focal length of the lens, the deviation is clearly $\partial = -y/f$, the lens being supposed collective.

This expression will therefore represent the deviation caused by the lens in *any* incident ray.

Now suppose that there are n thin lenses whose focal lengths are, respectively, $f_1, f_2 \ldots f_n$, arranged at intervals $a_1, a_2, \ldots a_{n-1}$ along an axis. For brevity, let $k = -1/f$, for all suffixes. Let any ray originally parallel to the axis strike the lenses in succession at distances $y_1, y_2 \ldots y_n$ from the axis, and let $\partial_1, \partial_2, \ldots \partial_n$ be the total deviations of the ray, after passing through the several lenses. Then, using the value of the deviation just given, and expressing the distances $y_2, y_3 \ldots$ in terms of the deviations, we obtain the equations

$$\begin{aligned}
\partial_1 &= k_1 y_1, \\
y_2 &= a_1 \partial_1 + y_1, \\
\partial_2 &= k_2 y_2 + \partial_1, \\
y_3 &= a_2 \partial_2 + y_2, \\
&\ldots\ldots\ldots\ldots\ldots \\
\partial_n &= k_n y_n + \partial_{n-1}.
\end{aligned}$$

From these equations it is easy to see that ∂_n is the numerator of the last convergent of the continued fraction

$$\frac{y_1}{1+}\,\frac{1}{k_1+}\,\frac{1}{a_1+}\,\frac{1}{k_2+}\ldots\ldots+\frac{1}{k_n}.$$

If F_n be the focal length of the equivalent lens, $\partial_n = -y_1/F_n = y_1 K_n$, say. Then K_n is equal to the numerator of the last convergent of the continued fraction

$$\frac{1}{1+}\,\frac{1}{k_1+}\,\frac{1}{a_1+}\,\frac{1}{k_2+}\ldots\ldots+\frac{1}{k_n}.$$

The values of the first few numerators are

$$1,\ k_1,\ a_1k_1+1,\ a_1k_1k_2+k_2+k_1,\ a_1a_2k_1k_2+a_2(k_1+k_2)+a_1k_1+1,$$
$$a_1a_2k_1k_2k_3+a_2k_3(k_1+k_2)+a_1k_1(k_2+k_3)+k_1+k_2+k_3,$$

from which we deduce the values

$$\frac{1}{F_2}=\frac{1}{f_1}+\frac{1}{f_2}-\frac{a_1}{f_1f_2},$$

$$\frac{1}{F_3}=\frac{1}{f_1}+\frac{1}{f_2}+\frac{1}{f_3}-\frac{a_1}{f_1}\left(\frac{1}{f_2}+\frac{1}{f_3}\right)-\frac{a_2}{f_3}\left(\frac{1}{f_1}+\frac{1}{f_2}\right)+\frac{a_1a_2}{f_1f_2f_3}.$$

These results might also have been obtained directly from the equations.

84. We may find a formula connecting two consecutive terms of the series K_1, K_2...... which gives a ready method of calculating their values. For the last two equations are

$$\partial_n = k_n y_n + \partial_{n-1},$$
$$y_n = a_{n-1}\,\partial_{n-1} + y_{n-1}.$$

Eliminating y_n between these equations, we deduce

$$\partial_n = (1 + a_{n-1}\,k_n)\,\partial_{n-1} + k_n\,y_{n-1}$$

or
$$\partial_n = (1 + a_{n-1}\,k_n)\,\partial_{n-1} + k_n\,\frac{d\partial_{n-1}}{dk_{n-1}}.$$

If now we substitute $\partial = Ky$, for all suffixes of ∂ and K, we arrive at the equation

$$K_n = (1 + a_{n-1}k_n)\,K_{n-1} + k_n\frac{dK_{n-1}}{dk_{n-1}},$$

which determines K_n as soon as K_{n-1} is known. For example,

$$K_4 = (1 + a_3k_4)\,\{k_1 + k_2 + k_3 + a_2k_3\,(k_1 + k_2) + a_1k_1\,(k_2 + k_3) + a_1a_2k_1k_2k_3\}$$
$$\quad + k_4\,\{1 + a_2\,(k_1 + k_2) + a_1k_1 + a_1a_2k_1k_2\},$$

which is equivalent to

$$\frac{1}{F_4} = \frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3} + \frac{1}{f_4} - \frac{a_1}{f_1}\left(\frac{1}{f_2} + \frac{1}{f_3} + \frac{1}{f_4}\right) - a_2\left(\frac{1}{f_1} + \frac{1}{f_2}\right)\left(\frac{1}{f_3} + \frac{1}{f_4}\right)$$
$$- \frac{a_3}{f_4}\left(\frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3}\right) + \frac{a_2a_3}{f_3f_4}\left(\frac{1}{f_1} + \frac{1}{f_2}\right) + \frac{a_3a_1}{f_4f_1}\left(\frac{1}{f_2} + \frac{1}{f_3}\right) + \frac{a_1a_2}{f_1f_2}\left(\frac{1}{f_3} + \frac{1}{f_4}\right)$$
$$- \frac{a_1a_2a_3}{f_1f_2f_3f_4}.$$

This is the value of the power of a lens equivalent to four given lenses, separated by intervals a_1, a_2, a_3 from each other.

Ex. Show that

$$\frac{1}{F_n} = -\begin{vmatrix} -k_n & -1 & 0 & 0 & \cdots\cdots & 0 & 0 \\ 1 & -a_{n-1} & -1 & 0 & \cdots\cdots & 0 & 0 \\ 0 & 1 & -k_{n-1} & 1 & \cdots\cdots & 0 & 0 \\ 0 & 0 & 1 & -a_{n-2} & \cdots\cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots\cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots\cdots & -a_1 & -1 \\ 0 & 0 & 0 & 0 & \cdots\cdots & 1 & -k_1 \end{vmatrix}$$

the determinant having $(2n-1)$ rows.

85. If the incident pencil be of any form, the position of the equivalent lens is not immaterial, and must be found.

Let the incident ray make an angle ∂ with the axis; then using the same notation as before, all the equations remain the same except the first, which is

$$\partial_1 = k_1 y_1 + \partial = y_1 \left(k_1 + \frac{\partial}{y_1}\right)$$

and therefore the final value of ∂_n will be the same as before, with $k_1 + \partial/y_1$ written for k_1. If the reciprocal of the focal length of the equivalent lens be denoted by K, since K involves k_1 only in the first degree, the new value of K will be

$$K' = K + \frac{\partial}{y_1}\frac{dK}{dk_1};$$

so that
$$\partial_n = Ky_1 + \partial \frac{dK}{dk_1}.$$

Let the distance of the equivalent lens behind the first lens of the system be x; then the incident ray will meet the lens at a distance from the axis equal to $y_1 + x\partial$, and therefore the inclination of the ray to the axis after refraction through it will be

$$\begin{aligned}\partial' &= K(y_1 + x\partial) + \partial \\ &= Ky_1 + \partial(1 + Kx).\end{aligned}$$

Equating this value to the inclination ∂_n, we get

$$1 + Kx = \frac{dK}{dk_1},$$

so that
$$x = \frac{1}{K}\left(\frac{dK}{dk_1} - 1\right).$$

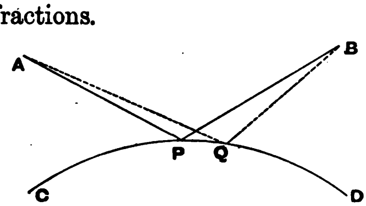
This determines the position of the lens so that it may be equivalent to the given system of lenses.

CHAPTER VI.

General Theorems. Caustics.

86. If a ray of light pass from a point A to another point B, through any number of media, undergoing any number of reflexions and refractions, then the actual laws of reflexion and refraction are such as to make $\Sigma(\mu\rho)$ a minimum, where ρ represents the length of the path of the ray situated in the medium whose refractive index is μ. Conversely if we assume the path of light to be such as to make $\Sigma\mu\rho$ a minimum, we are led to the actual laws of reflexion and refraction. The expression $\Sigma\mu\rho$ is frequently called the *reduced path*.

We shall first prove this general theorem for a single reflexion and a single refraction, and afterwards extend it to any number of reflexions and refractions.

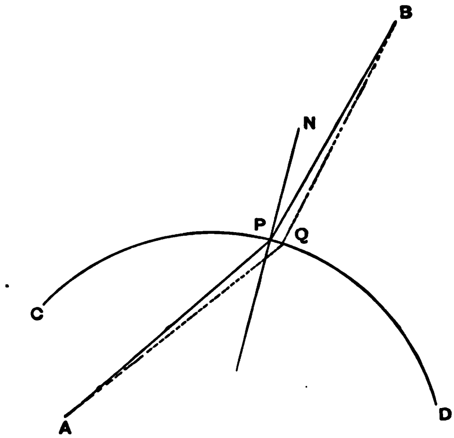


Let APB be the path of a ray of light which travels in a homogeneous medium from a point A to a point B, undergoing one reflexion at a surface CD; then the total path between A and B is a minimum, that is, $AP+PB$ is less along the actual path than along any consecutive path as AQB.

For a variation of P perpendicular to the plane APB, this proposition is clearly true. Let AQB be a consecutive path in the plane APB. Then the difference $AQ-AP$ is equal to the projection of PQ on AP; and similarly the difference $BP-BQ$ is

equal to the projection of PQ on PB. But these projections are equal, because AP and PB are equally inclined to PQ. Thus, $AQ + QB = (AP + PB)$, which proves that the increment of the total path vanishes, and therefore the total path is a minimum.

A similar theorem holds if we take the path from A to B, supposing the ray to suffer a refraction at a surface CD. Let μ, μ' be the refractive indices of the two media, then $\mu AP + \mu' PB$ is a minimum for the actual path.



Draw the normal PN, and let the angles of incidence and refraction be ϕ, ϕ'; then $\mu \sin\phi = \mu' \sin\phi'$. Let AQB be a consecutive path; it will be sufficient to take the case when Q is in the plane APB.

Then $$\mu AQ - \mu AP = \mu PQ \sin\phi$$

and $$\mu' BP - \mu' BQ = \mu' PQ \sin\phi'.$$

Hence the whole variation,

$$\begin{aligned} \mu AQ + \mu' BQ - (\mu AP + \mu' BP) & \\ &= PQ(\mu \sin\phi - \mu' \sin\phi') \\ &= 0. \end{aligned}$$

This shows that for the actual path, $\mu AP + \mu' PB$ is a minimum.

The previous theorem is a particular case of this; we have only to put $\mu' = -\mu$ to deduce it from the more general theorem.

Next, suppose that the ray of light in its passage from A to B undergoes any number of refractions or reflexions. Let ρ be the

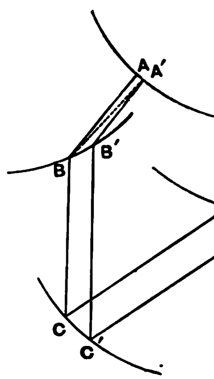
length of the path in any medium whose refractive index is μ. Then it has been shown that $\Sigma\mu\rho$ is a minimum for separate variations of the points of incidence between consecutive media; and therefore by the principle of superposition of small variations, it will be a minimum when simultaneous variations are admitted. The actual path, therefore, makes $\Sigma\mu\rho$ a minimum between any two points. If the variations of refractive index be gradual, the same principle holds good, and the path of the ray of light is such that $\int\mu ds$ is a minimum.

87. Another important proposition, enunciated by Malus, easily follows from the preceding.

Any system of rays originally normal to a surface, will always retain the property of being normal to a surface after any number of reflexions or refractions.

The general theory of systems of lines will be given later; but we may remark here that a doubly infinite system of lines is not in general a system of normals.

Let $ABCDE$, $A'B'C'D'E'$... be a series of rays normal to a sur-



face at A, which undergo any number of refractions and reflexions. Measure off along these rays distances to E, E'......, such that $\Sigma\mu\rho$ is the same along each ray; then we shall show that the rays are finally normal to a surface EE'. Join $A'B$ and $E'D$. Then $\Sigma\mu\rho$ along $ABCDE$ is the same as along $A'B'C'D'E'$. But by what has been shown above for any ray and its consecutive it follows that $\Sigma\mu\rho$ along $A'BCDE'$ is the same as along $A'B'C'D'E'$, and therefore the same as along $ABCDE$. Take away the common

parts; then if μ, μ' belong to the final media, there remains the equation, $\mu A'B + \mu' E'D = \mu AB + \mu' DE$. But, since AB is normal to the surface AA', $A'B = AB$ ultimately, and therefore, $DE' = DE$; that is, EE' is perpendicular to DE. The same may be proved for every point E' near E, and thus the surface EE' near E is perpendicular to the ray DE, and by similar reasoning to every other ray of the system.

88. These theorems may also be proved analytically.

Let a ray of light pass through several media of different refractive indices. Let (α, β, γ) be the direction cosines of the incident ray, and (x, y, z) a point on it; then (α, β, γ) may be regarded as known functions of x, y, z. Let this ray meet the first surface in the point (ξ_1, η_1, ζ_1), and let $(\alpha_2, \beta_2, \gamma_2)$ be the direction cosines of the refracted ray in the second medium and (x_2, y_2, z_2) be a point on the refracted ray. Then at the first refraction the direction cosines of the rays are connected by the equations

$$\left.\begin{aligned} \mu\alpha - \mu_2\alpha_2 &= (\mu \cos\phi - \mu_2 \cos\phi_2)\,\lambda_1 \\ \mu\beta - \mu_2\beta_2 &= (\mu \cos\phi - \mu_2 \cos\phi_2)\,\mu_1 \\ \mu\gamma - \mu_2\gamma_2 &= (\mu \cos\phi - \mu_2 \cos\phi_2)\,\mu_1 \end{aligned}\right\}$$

where $(\lambda_1, \mu_1, \nu_1)$ are the direction cosines of the normal to the surface.

Also, $$\lambda_1 d\xi_1 + \mu_1 d\eta_1 + \nu_1 d\zeta_1 = 0,$$
and therefore
$$(\mu\alpha - \mu_2\alpha_2)\, d\xi_1 + (\mu\beta - \mu_2\beta_2)\, d\eta_1 + (\mu\gamma - \mu_2\gamma_2)\, d\zeta_1 = 0.$$

Let r_1 denote the distance between (x, y, z) and (ξ_1, η_1, ζ_1) and r_1' the distance between (ξ_1, η_1, ζ_1) and (x_2, y_2, z_2);

then $$\left.\begin{aligned} r_1^2 &= (\xi_1 - x)^2 + (\eta_1 - y)^2 + (\zeta_1 - z)^2 \\ r_1'^2 &= (x_2 - \xi_1)^2 + (y_2 - \eta_1)^2 + (z_2 - \zeta_1)^2 \end{aligned}\right\}.$$

Also $$\left.\begin{aligned} \alpha &= (\xi_1 - x)/r_1 \\ \beta &= (\eta_1 - y)/r_1 \\ \gamma &= (\zeta_1 - z)/r_1 \end{aligned}\right\}, \qquad \left.\begin{aligned} \alpha_2 &= (x_2 - \xi_1)/r_1' \\ \beta_2 &= (y_2 - \eta_1)/r_1' \\ \gamma_2 &= (z_2 - \zeta_1)/r_1' \end{aligned}\right\}.$$

If therefore we differentiate the values of r_1^2, and $r_1'^2$, we find

$$\left.\begin{aligned} dr_1 &= \alpha\,(d\xi_1 - dx_1) + \beta\,(d\eta_1 - dy_1) + \gamma\,(d\zeta_1 - dz_1) \\ dr_1' &= \alpha_1\,(dx_2 - d\xi_1) + \beta_1\,(dy_2 - d\eta_1) + \gamma_1\,(dz_2 - d\zeta_1) \end{aligned}\right\}.$$

From these values, by virtue of the previous equation connecting $d\xi_1$, $d\eta_1$, $d\zeta_1$, we deduce the equation

$$\mu dr_1 + \mu_2 dr_1' = \mu_2 (\alpha_2 dx_2 + \beta_2 dy_2 + \gamma_2 dz_2) - \mu (\alpha dx + \beta dy + \gamma dz).$$

Also let r_2 be the distance from the point (x_2, y_2, z_2) to the next surface and r_2' the distance from the surface to a point (x_3, y_3, z_3), on the refracted ray in the next medium. Then, as before,

$$\mu_2 dr_2 + \mu_3 dr_2' = \mu_3 (\alpha_3 dx_3 + \beta_3 dy_3 + \gamma_3 dz_3) - \mu_2 (\alpha_2 dx_2 + \beta_2 dy_2 + \gamma_2 dz_2),$$

and so on. If, therefore, ρ, $\rho_2, \ldots\ldots \rho_{n-1}$, ρ' be the whole lengths of the paths in the different media, we find by adding all the equations similar to the last two,

$$\mu d\rho + \mu_2 d\rho_2 + \ldots\ldots + \mu' d\rho'$$
$$= \mu' (\alpha' dx' + \beta' dy' + \gamma' dz') - \mu (\alpha dx + \beta dy + \gamma dz).$$

If $\mu (\alpha dx + \beta dy + \gamma dz)$ be a perfect differential it appears that $\mu' (\alpha' dx' + \beta' dy' + \gamma' dz')$ is a perfect differential also; in other words, *if the rays are normal to a surface at any time, they will always be normal to a surface.*

Let $$\mu (\alpha dx + \beta dy + \gamma dz) = dV,$$
and $$\mu' (\alpha' dx' + \beta' dy' + \gamma' dz') = dV',$$
then, by integration, $$V' - V = \Sigma (\mu\rho).$$

Thus *the reduced path from one surface to the other is the same for all rays of the system.*

When the initial and final points are fixed, dx, dy, dz and dx', dy', dz' all vanish, so that $\Sigma (\mu d\rho) = 0$, which proves that $\Sigma (\mu\rho)$, or *the reduced path, is a minimum.*

The function V is called *the Characteristic Function* of the System.

89. From this we can pass to any heterogeneous medium. The path will be such as to make $\int \mu ds$ or V a minimum. The rays will all be normal to the surface $V =$ constant. When we know V as a function of x, y, z at all points of space, in terms of the coordinates (x, y, z), we know the direction of the ray at any point.

For $$\frac{dV}{dx} = \mu\alpha, \quad \frac{dV}{dy} = \mu\beta, \quad \frac{dV}{dz} = \mu\gamma,$$

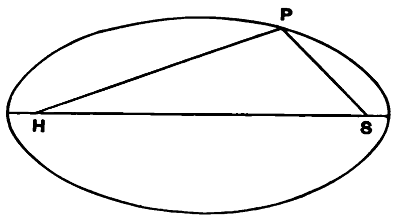
and therefore $$\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2 = \mu^2,$$

and these equations determine $\mu, \alpha, \beta, \gamma$.

90. A system of rays which can be cut at right angles by a surface, we shall call an *orthotomic* system.

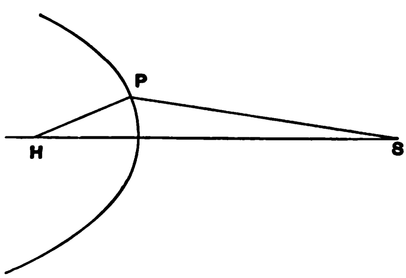
A system of rays diverging from a point, or such that by any combination of mirrors or refracting surfaces they can be made to meet in a point, is clearly orthotomic; for a sphere whose centre is the point through which all the rays pass, will cut them all at right angles.

If a system of rays diverging from a point converge to another point after any number of reflexions and refractions, the values of $\Sigma\mu\rho$ taken from one point to the other will be the same for all rays. Thus, in order to condense rays issuing from one point S, on a second point H, by means of a single reflexion at a curved surface, we choose our surface such that $SP + PH$ may be the same for all paths, and therefore the surface must be an ellipsoid of revolution whose foci are S and H.



If the rays are parallel, the point S will be at infinity, and the surface is a paraboloid of revolution whose axis is parallel to the common direction of the rays.

Next, let us find the form of the surface which will refract



to a point H all the rays proceeding from a point S. Let μ, μ'

be the refractive indices of the media; then if P be any point of the surface, the surface must be such that

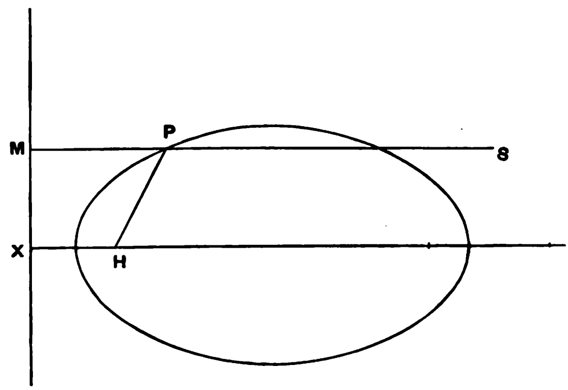
$$\mu SP + \mu' HP = c,$$

where c is a constant.

Hence the surface is formed by the revolution of a Cartesian oval of which S and H are foci. The theory of the Cartesian oval may be found in Williamson's *Differential Calculus*, Appendix.

As a particular case suppose the rays parallel, so that S is at infinity. Draw a plane MX perpendicular to the rays, and let any ray be produced to meet this surface in M. Then

$$\mu SP + \mu' HP = c.$$

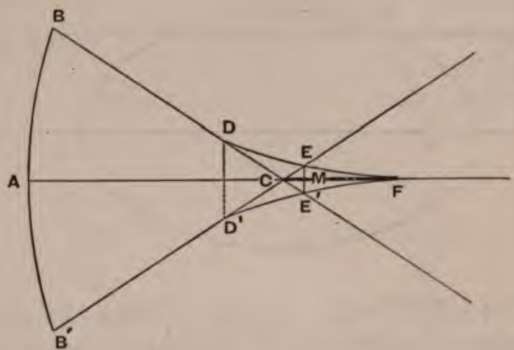


But $\mu SP + \mu PM$ is also constant. Choose the plane MX so that this constant quantity may be equal to c; then $\mu' HP = \mu PM$, and therefore the surface is formed by the revolution of a conic whose focus is H and directrix MX, about its major axis.

91. In general, consecutive rays do not intersect. But if we take consecutive rays of an orthotomic system which meet the orthogonal surface along a line of curvature, these rays will intersect each other, and will envelope a curve, called a *caustic* curve. Consecutive lines of curvature of the same system will give rise to consecutive caustic curves, and these curves will generate a caustic surface, to which every ray of the system is a tangent. Similarly, the other system of lines of curvature will determine another caustic surface. Thus every ray of the system will touch two caustic surfaces.

When the system of rays is symmetrical about an axis, the orthogonal surface is a surface of revolution. One set of lines of curvature are the meridian curves, and the caustic surface corresponding to them will be generated by the revolution about the axis of symmetry, of the evolute of the meridian curve. The other set of lines of curvature are circles whose centres lie on the axis. The rays which are normal along one of these circles will meet in a point on the axis, forming a cone of revolution. Thus the second caustic surface will consist of a portion of the axis of symmetry.

92. The character of a limited pencil of rays is shown in the figure; BAB' is the orthogonal surface, F is the cusp of the caustic curve.



If the pencil be received on a screen perpendicular to the axis, the nature of the caustic surface can be shown by examining the bright patch of light on the screen as the screen is moved from DD' towards F. At DD', there will be a circular patch of light with a brighter ring round its outer edge, and as the screen is moved along, this ring will gradually contract. As soon as C is reached, the other part of the caustic surface is shown, and a bright spot is developed in the centre. When the screen is at EE' the circle of light reaches its minimum; this circle is called the *least circle of aberration*. When this position is passed, the outer boundary expands again though the bright ring still contracts. Beyond F, no part of the screen is specially illuminated.

93. If any ray BCE meet the axis in C, then FC is called the *longitudinal aberration* of this ray. By symmetry, FC is an even function of the inclination of the ray to the axis; and if the pencil be small, we may take it as varying as the square of the inclination. Thus let m denote the tangent of the inclination of the ray to the axis, then we may suppose the longitudinal aberration to be given by the equation $FC = cm^2$, powers of m beyond the third being neglected. Take F as origin and FA as the axis of x. Then the equation to the ray is

$$y = m(x - cm^2).$$

If we find the envelope of this line, regarding m as a variable parameter, this envelope will be the caustic curve in the neighbourhood of F. Differentiating with respect to m, we get

$$0 = x - 3cm^2,$$

and therefore eliminating m, the equation to the envelope is found to be

$$27cy^2 = 4x^3.$$

From this equation it appears that *the form of the caustic curve near the cusp is a semi-cubical parabola.*

94. To find the magnitude and position of the least circle of aberration, we must find the intersection of any ray with the extreme ray; then by making the ordinate of the point of intersection a minimum, we shall find the position and radius of the circle.

Let the equation to the extreme ray be

$$y = kx - ck^3,$$

and that of any other ray

$$y = mx - cm^3.$$

Then eliminating x, we find

$$y(m-k) = cmk(m^2 - k^2),$$

or

$$y = ck(m^2 + km);$$

that is

$$y = ck\left\{\left(m + \frac{k}{2}\right)^2 - \frac{k^2}{4}\right\}.$$

The value of y is therefore a minimum when $m = -\frac{1}{2}k$, and the radius of the least circle of aberration is given by the equation

$$y = -\tfrac{1}{4}ck^3 = -r, \text{ say.}$$

The corresponding value of x is found from the equation

$$y = kx - ck^3$$

to be

$$x = \tfrac{3}{4}ck^2.$$

Thus *the distance of the centre of the least circle of aberration from the cusp is three-fourths of the longitudinal aberration of the extreme ray.*

To find the lateral aberration of the extreme ray, we have only to put $x = 0$, in the equation of that ray; and then we get

$$y = -ck^3.$$

From this it appears that *the radius of the least circle of aberration is one-fourth of the lateral aberration of the extreme ray.*

95. If a mirror or a lens which has a small aberration still uncorrected, be used as a part of an optical instrument, we see from the above investigation that symmetrical pencils do not in general meet in a point, but a section of the pencil by a screen is a small circle. To see the image as clearly as possible the circle must be made as small as possible, and therefore the screen must have the position of the least circle of aberration. The image of an object as seen through the instrument will therefore not be distinct; it will consist of a series of small circular patches of light overlapping each other. This defect is not so great however as might be at first imagined, for the least circle of aberration is not uniformly bright, but it is brightest in the centre and the brightness decreases rapidly towards the edges, and the image of a point is reduced almost to the centre itself, when the incident light is feeble.

To show this, we may consider an approximate theory in a simple case. With the same notation as before, let us find the aperture of the pencil at the orthogonal surface; let η be the radius of this aperture, and a the abscissa of the apex of the surface; then,

$$\eta = ka - ck^3, \text{ approximately,}$$
$$= ka, \text{ nearly.}$$

The corresponding radius of the least circle of aberration is $y = \tfrac{1}{4}ck^3$.

Let us consider the light which is included between the circles whose radii are η and $\eta + d\eta$; the area of the zone will be $2\pi\eta d\eta$.

The corresponding zone on the screen will be $2\pi y dy$. Supposing that the amount of light emerging from the orthogonal surface is proportional to the area of the zone on that surface, then the brightness of the small zone of the least circle of aberration will be proportional to $2\pi\eta d\eta/2\pi y dy$, that is, to

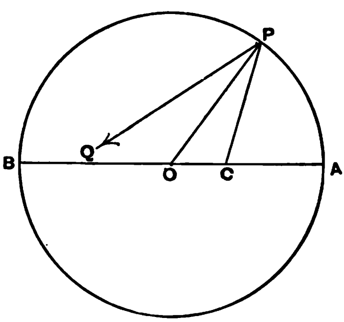
$$\frac{a^2 k dk}{\frac{1}{4}ck^3 \cdot \frac{3}{4}ck^2 dk}, \text{ or } \frac{16}{3}\frac{a^2}{c^2} \cdot \frac{1}{k^4}.$$

Now y varies as k^3, and therefore the brightness of the least circle of aberration varies inversely as $y^{\frac{4}{3}}$; this shows that it decreases from the centre towards the circumference.

This investigation is only a rough approximation to the truth, but it serves to show how the images preserve their distinctness even when a small aberration exists.

96. We shall now investigate the equations and properties of the caustic curves for systems of rays issuing from a point and reflected or refracted at a surface, in a few of the most interesting and simple cases.

The equation to the caustic by reflexion at a circle of rays proceeding from a luminous point has been found by Lagrange in the following manner:



Let C be the luminous point, and AOB that diameter of the reflecting circle which passes through C; any ray CP incident on the circle at P will be reflected in the direction PQ, so that CP and PQ make equal angles with the radius OP. Let the angle AOP be denoted by α, and let $OC = c = 1/p$, and $OA = a = 1/k$. Then the equations to the incident and reflected rays may be written in

the forms

$$u = A\cos\theta + B\sin\theta,$$
$$u = A\cos(2\alpha - \theta) + B\sin(2\alpha - \theta).$$

For if we draw radii vectores making equal angles with OP, their vectorial angles θ and θ' will be such as to make $\theta + \theta' = 2\alpha$, and therefore the corresponding values of u given by these two equations are equal. We must find the constants A, B from the fact that the incident ray passes through the points C and P. Putting $\theta = 0$, and $\theta = \alpha$, respectively, for the points C and P, the equation to CP gives us

$$p = A$$
$$k = A\cos\alpha + B\sin\alpha.$$

If we substitute the values of A and B into the equation to the reflected ray it becomes

$$u\sin\alpha = k\sin(2\alpha - \theta) - p\sin(\alpha - \theta).$$

Let $2\alpha - \theta = 2\phi$, so that $\alpha = \phi + \frac{1}{2}\theta$; then this equation may be written

$$u\sin(\phi + \tfrac{1}{2}\theta) + p\sin(\phi - \tfrac{1}{2}\theta) = k\sin 2\phi;$$

that is
$$\frac{P}{\cos\phi} + \frac{Q}{\sin\phi} = 1,$$

where
$$P = \frac{(u+p)\cos\frac{1}{2}\theta}{2k},$$

$$Q = \frac{(u-p)\sin\frac{1}{2}\theta}{2k}.$$

The arbitrary parameter α enters into this equation only through ϕ. To find the envelope of the reflected ray, we equate to zero the first differential of the equation with regard to ϕ, which gives

$$\frac{P}{\cos^3\phi} = \frac{Q}{\sin^3\phi} = \lambda, \text{ say};$$

then if we substitute the values of P and Q in the equation of the ray, we find that $\lambda = 1$. Eliminating ϕ, the equation to the envelope is

$$P^{\frac{2}{3}} + Q^{\frac{2}{3}} = 1.$$

Expressed in polar coordinates it is

$$\{(u+p)\cos\tfrac{1}{2}\theta\}^{\frac{2}{3}} + \{(u-p)\sin\tfrac{1}{2}\theta\}^{\frac{2}{3}} = (2k)^{\frac{2}{3}}.$$

To rationalise the equation $P^{\frac{2}{3}} + Q^{\frac{2}{3}} = 1$, we cube both sides; it becomes

$$P^2 + Q^2 + 3P^{\frac{2}{3}}Q^{\frac{2}{3}}(P^{\frac{2}{3}} + Q^{\frac{2}{3}}) = 1,$$

or

$$1 - P^2 - Q^2 = 3P^{\frac{2}{3}}Q^{\frac{2}{3}}.$$

Cubing the equation again, we get finally,

$$(1 - P^2 - Q^2)^3 = 27P^2Q^2.$$

The equation may easily be transformed into Cartesian coordinates. For

$$P = \frac{a}{2}\left(\frac{1}{r} + \frac{1}{c}\right)\cos\tfrac{1}{2}\theta,$$

$$Q = \frac{a}{2}\left(\frac{1}{r} - \frac{1}{c}\right)\sin\tfrac{1}{2}\theta,$$

and therefore

$$P^2 + Q^2 = \frac{a^2}{4}\left\{\frac{1}{r^2} + \frac{1}{c^2} + \frac{2}{rc}\cos\theta\right\}$$

$$= \frac{a^2}{4c^2r^2}\{r^2 + c^2 + 2cx\}.$$

Also,

$$PQ = \frac{a^2}{8r^2c^2}(c^2 - r^2)\sin\theta$$

$$= \frac{a^2}{8r^3c^2}(c^2 - r^2)\,y.$$

Hence the equation to the caustic becomes

$$\{(4c^2 - a^2)(x^2 + y^2) - 2a^2cx - a^2c^2\}^3 = 27a^4c^2y^2(x^2 + y^2 - c^2)^2.$$

97. If in the equation to the caustic we put $\theta = 0$, and therefore $Q = 0$, $P = (u+p)/2k$, it becomes

$$(1 - P^2)^3 = 0,$$

and therefore

$$u + p = \pm 2k,$$

and each of these points is a *triple* intersection. Expressed in terms of a and c, the distances of these points from the centre of the circle, are

$$r = \frac{ac}{2c - a}, \quad r = -\frac{ac}{2c + a}.$$

These points are *cusps* on the caustic.

Again, if we wish to find the points of intersection of the caustic with the circle $u = p$, we get $Q = 0$ and $P = p/k \cos \frac{1}{2}\theta$, and the equation to the caustic gives

$$(1 - P^2)^3 = 0,$$

which reduces to $$p \cos \tfrac{1}{2}\theta = \pm k,$$

or $$a \cos \tfrac{1}{2}\theta = \pm c.$$

Each of these points is a *triple* point of intersection; *the reflected ray, therefore, touches the circle* $r = c$, *and the corresponding incident ray is perpendicular to OC.* These triple points of intersection are *cusps*, and the tangent at the cusp is perpendicular to the radius vector; they are imaginary if c is greater than a, that is, if the luminous point is outside the circular reflector.

98. To find the directions of the asymptotes, we must make $u = 0$, in the equation to the curve. The values of P and Q are then

$$P = \frac{p}{2k} \cos \tfrac{1}{2}\theta = \frac{a}{2c} \cos \tfrac{1}{2}\theta,$$

$$Q = -\frac{p}{2k} \sin \tfrac{1}{2}\theta = -\frac{a}{2c} \sin \tfrac{1}{2}\theta;$$

and from the equation to the caustic, we derive the equation

$$\left\{1 - \frac{a^2}{4c^2}\right\}^3 = 27 \frac{a^4}{64c^4} \sin^2 \theta,$$

that is, $$27a^4c^2 \sin^2 \theta = (4c^2 - a^2)^3.$$

This equation gives the directions of the asymptotes, and shows that they are imaginary if c be less than $\frac{1}{2}a$.

We shall now find the length of the perpendicular on them from the origin.

Differentiating the equation

$$P^{\frac{2}{3}} + Q^{\frac{2}{3}} = 1,$$

and afterwards putting $u = 0$, we get after dividing by common factors,

$$\frac{1}{(\cos \frac{1}{2}\theta)^{\frac{1}{3}}} \left\{\frac{du}{d\theta} \cos \tfrac{1}{2}\theta - \frac{p}{2} \sin \tfrac{1}{2}\theta\right\} - \frac{1}{(\sin \frac{1}{2}\theta)^{\frac{1}{3}}} \left\{\frac{du}{d\theta} \sin \tfrac{1}{2}\theta - \frac{p}{2} \cos \tfrac{1}{2}\theta\right\} = 0.$$

This gives

$$\frac{du}{d\theta}(\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta)^{\frac{1}{3}}\{\cos^{\frac{2}{3}}\tfrac{1}{2}\theta-\sin^{\frac{2}{3}}\tfrac{1}{2}\theta\}+\frac{p}{2}\{\cos^{\frac{4}{3}}\tfrac{1}{2}\theta-\sin^{\frac{4}{3}}\tfrac{1}{2}\theta\}=0,$$

or $$\frac{du}{d\theta}(\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta)^{\frac{1}{3}}=-\frac{1}{2c}(\cos^{\frac{2}{3}}\tfrac{1}{2}\theta+\sin^{\frac{2}{3}}\tfrac{1}{2}\theta)$$

$$=-\frac{1}{2c}\left(\frac{2c}{a}\right)^{\frac{2}{3}}\{P^{\frac{2}{3}}+Q^{\frac{2}{3}}\}$$

$$=-\frac{1}{2c}\left(\frac{2c}{a}\right)^{\frac{2}{3}}.$$

Hence $$\frac{du}{d\theta}(\sin\theta)^{\frac{1}{3}}=-\frac{1}{c}\left(\frac{c}{a}\right)^{\frac{2}{3}},$$

or $$\frac{du}{d\theta}\sqrt{\frac{4c^2-a^4}{3}}=-1.$$

If we denote the length of the perpendicular from the centre on the asymptote by ϖ, we get

$$\varpi=\sqrt{\frac{4c^2-a^2}{3}}.$$

The asymptotes are imaginary, if c be less than $\frac{1}{2}a$; and when $c=\frac{1}{2}a$, they coincide with the axis of x.

99. We shall next find the points of intersection of the caustic with the reflector; for this purpose we shall use the Cartesian equations. If we make $x^2+y^2=a^2$ in the equation to the caustic it becomes

$$(3a^2c^2-a^4-2a^2cx)^3=27a^4c^2(a^2-c^2)^2(a^2-x^2),$$

or, by expansion and division by a^4,

$$8a^2c^3x^3-c^2x^2(15a^4-18a^2c^2-27c^4)+6cx(3c^2-a^2)^2$$
$$+a^8+18a^6c^2-27a^4c^4=0,$$

which may be written in the form

$$(cx-a^2)^2\{8a^2cx+a^4+18a^2c^2-27c^4\}=0.$$

Hence the caustic *touches* the reflector at the points given by the equation $cx=a^2$, which are the points of contact of the tangents drawn from the luminous point to the reflector if the luminous point be outside the reflector, and are imaginary if this point be inside. The other point of intersection is determined by the equation

$$x=\frac{27c^4-18a^2c^2-a^4}{8a^2c}.$$

This value of x is numerically less or greater than a, according as c is greater or less than a; that is, according as the luminous point is outside or inside the reflecting circle.

The shapes of the caustic curves for different positions of the luminous points are shown in the following figures. In the first figure the incident rays are parallel; the other figures represent the caustic curve as the luminous point approaches nearer and nearer to the centre of the reflecting circle.

Fig. 1. $c=\infty$.

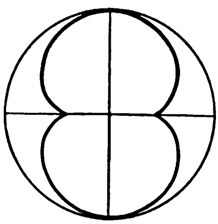


Fig. 2. $c>a$.

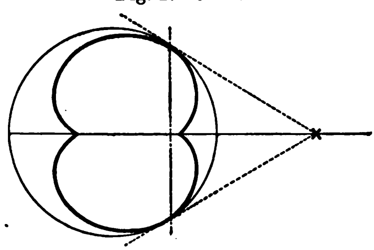


Fig. 3. $c=a$.

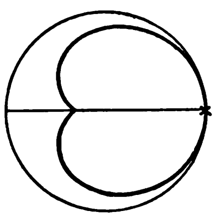


Fig. 4. $c<a, >\frac{1}{2}a$.

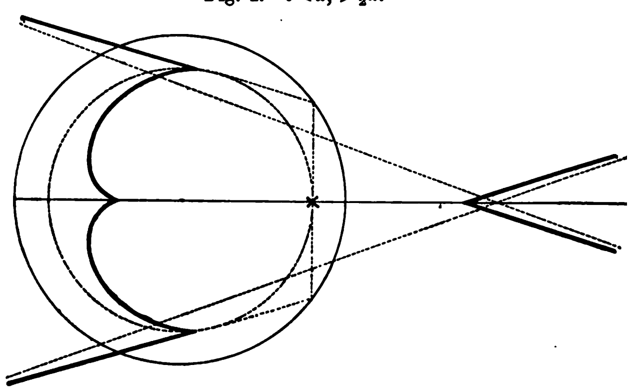


Fig. 5. $c=\frac{1}{2}a$.

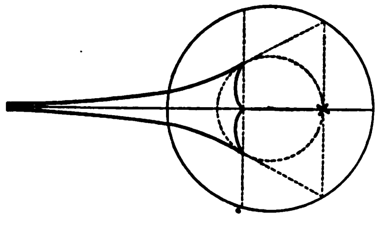
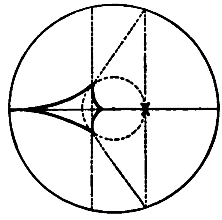
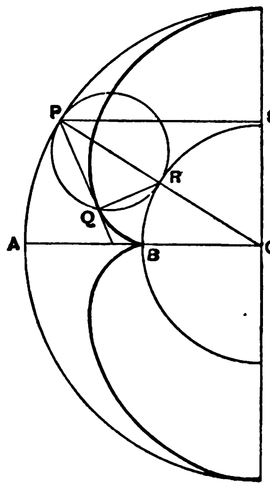


Fig. 6. $c=\frac{1}{3}a$.



100. The caustic by reflexion at a circle may be found by elementary geometry in two cases, first, when the incident rays are parallel, and secondly, when they diverge from a point on the circumference of the circle.

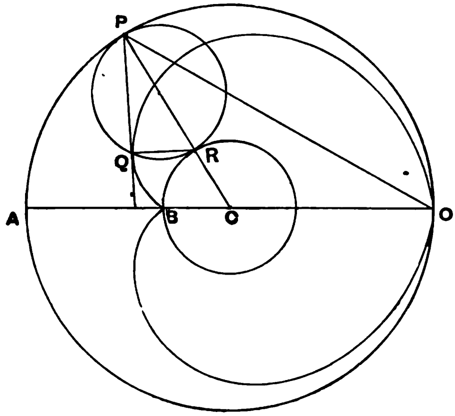
When the incident rays are parallel, the caustic is an epicycloid formed by the rolling of one circle upon another of twice its radius.



For from the centre C of the reflecting circle, draw the radius CA parallel to the incident rays; then the caustic is symmetrical with regard to the line CA. Let SP be any one of the incident rays, reflected by the circle at the point P in the direction PQ. Join CP; then by the law of reflexion, CP will bisect the angle SPQ. With centre C and a radius equal to half the radius of the given circle, describe the circle BR bisecting the radii CA, CP in B, R, respectively. On PR as diameter describe another circle meeting the reflected ray in Q, and join QR. Since SP is parallel to CB, the angle SPC is equal to the angle PCB; and therefore the angle QPR is equal to the angle RCB. The angle QPR is subtended at the circumference of the circle by an arc QR; and the angle RCB is subtended at the centre of the other circle by the arc RB, and the radius of the second circle is double the radius of the first, and therefore the arc QR is equal to the arc RB; and if the circle PQR were to roll along the circle RB, the

point Q would finally coincide with B. Now as Q begins to move, the point of contact R is for an instant fixed, so that the motion of Q is perpendicular to QR; and therefore the reflected ray PQ touches the curve described by Q. This is true whatever the position of the point P. The locus of Q is an epicycloid, and this is the caustic curve required.

101. *If the incident rays diverge from a point in the circumference of the reflecting circle, the caustic curve is a cardioid,* or, in other words, the caustic may be described as an epicycloid in which the rolling circle is equal to the fixed circle.

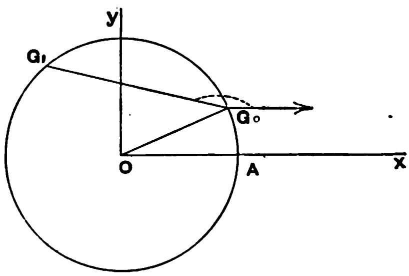


Let O be the origin of the incident rays, OCA the diameter of the reflecting circle; then the caustic curve will be symmetrical about the line OCA. Let OP be any incident ray which is reflected at P by the circle in the direction PQ. Join CP; then by the law of reflexion, CP will bisect the angle OPQ. With centre C and radius equal to one-third of the radius of the given circle, describe a circle meeting CA and CP in B and R, respectively, and on PR as diameter describe another circle cutting the reflected ray in Q; join QR. The radii of the two smaller circles will be equal to each other. Now, since the triangle CPO is isosceles, the external angle PCB is double of the angle CPO, and therefore double of the angle QPR. Hence the arcs RB, QR subtend equal angles at the centres of their respective circles, and therefore these arcs are equal. If the circle PQR were to roll along the circle RB, the point Q would finally come to B. As the circle PQR begins

to roll, the point of contact R is for a moment stationary, and therefore Q begins to move perpendicular to QR along PQ. From this it follows that the reflected ray touches the curve described by the point Q. This is true whatever the position of the point P. The locus of Q is a cardioid, and this is the caustic required.

102. There are two cases in which we can find the caustic after the rays have been reflected at a circle any number of times; first, when the incident rays are parallel, and secondly, when they diverge from a point in the circumference.

Let a ray be reflected any number of times at a circle; and let G_0G_1 be the first path of the ray across the circle, making an angle ψ_0 with the positive direction of the axis of x, and let the angle G_0OA be denoted by θ_0. Let θ, ψ be corresponding



angles for the nth reflected ray. Then the equation to this ray will be

$$y - c\sin\theta = \tan\psi\,(x - c\cos\theta),$$

or

$$y\cos\psi - x\sin\psi + c\sin(\psi - \theta) = 0,$$

where c is the radius of the circle.

But if the angle OG_0G_1 be denoted by ϕ, we have

$$\theta = \theta_0 + n(\pi - 2\phi) = \theta_0 + n\pi - 2n\phi$$

and

$$\psi = \psi_0 + n(2\pi - 2\phi) = \psi_0 + 2n\pi - 2n\phi.$$

Hence the equation to the nth reflected ray becomes

$$x\sin(\psi_0 - 2n\phi) - y\cos(\psi_0 - 2n\phi) = (-1)^n c\sin(\psi_0 - \theta_0).$$

First, let the rays be incident parallel to the axis of x; then

we may write $\theta_0 = \phi$, $\psi_0 = \pi$, and the equation to the reflected ray is

$$x \sin 2n\phi + y \cos 2n\phi = (-1)^n c \sin \phi.$$

To find the envelope of this line, we must differentiate the equation with respect to the parameter ϕ; we thus get the equation

$$x \cos 2n\phi - y \sin 2n\phi = (-1)^n \frac{1}{2n} c \cos \phi.$$

This equation, combined with the equation to the ray, determines the caustic.

If we solve the equations, we find that any point on the caustic may be represented by the coordinates

$$x = (-1)^n \frac{c}{4n} \{(2n+1) \cos (2n-1) \phi - (2n-1) \cos (2n+1) \phi\},$$

$$y = (-1)^n \frac{c}{4n} \{-(2n+1) \sin (2n-1) \phi + (2n-1) \sin (2n+1) \phi\}.$$

But the equation to an epicycloid in which the radius of the fixed circle is a, and that of the rolling circle b, is

$$x = (a+b) \cos \theta - b \cos \frac{a+b}{\theta} \theta.$$

$$y = (a+b) \sin \theta - b \sin \frac{a+b}{\theta} \theta.$$

The forms of these equations are the same, if instead of θ we write $-(2n-1)\phi$. Also, comparing the equations in order that they may be identical we must further have

$$a + b = c \frac{2n+1}{4n},$$

$$b = c \frac{2n-1}{4n},$$

and therefore $$a = c \frac{1}{2n}.$$

The caustic is therefore an epicycloid. When n is even, the cusp is on the axis of x on the positive side of the origin. When n is odd, it is necessary to change the signs of x and y, and therefore the epicycloid points the opposite way, the cusp being on the negative side of the origin.

103. Next, let the rays diverge from the point A on the circumference. Then $\theta_0 = 0$, $\psi_0 = \pi - \phi$, and the equation to the reflected ray is

$$x \sin (2n+1) \phi + y \cos (2n+1) \phi = (-1)^n c \sin \phi.$$

The envelope of this line may be found as before. Differentiating with respect to the variable parameter, we get the equation

$$x \cos (2n+1) \phi - y \sin (2n+1) \phi = (-1)^n \frac{1}{2n+1} c \cos \phi ;$$

and these two equations give

$$x = \frac{(-1)^n c}{2n+1} \{(n+1) \cos 2n\phi - n \cos (2n+2) \phi\},$$

$$y = \frac{(-1)^n c}{(2n+1)} \{-(n+1) \sin 2\phi + n \sin (2n+2) \phi\},$$

which represent the coordinates of any point on the caustic.

This is again an *epicycloid*, the radii of the fixed and rolling circles being, respectively,

$$a = \frac{c}{2n+1}, \quad b = \frac{n}{2n+1} c.$$

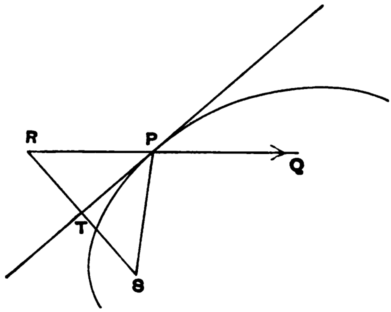
When n is even, the cusp is on the positive side of the origin, and when n is odd, it is on the negative side.

In the case in which n is unity, the values of a and b become equal, and the epicycloid becomes a cardioid.

104. In general, as we have seen, the reflected or refracted rays are the normals to a series of curves, which are sometimes called *secondary caustics*; any one of these has the reflected or refracted rays for normals and consequently the caustic curve for evolute. It is usually easier to find a secondary caustic than the caustic itself; for instance, for rays refracted at a straight line a secondary caustic is an ellipse, and for rays refracted at a circle, the Cartesian.

There are very convenient constructions for determining secondary caustics for rays issuing from a point and reflected or refracted at a curve.

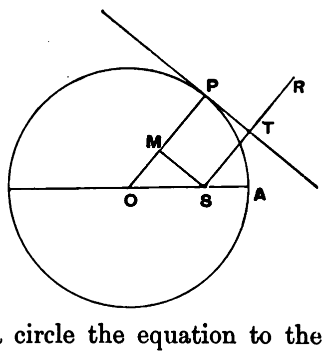
Let PT be the tangent at any point of a curve, and let S be the bright point. Draw ST perpendicular to the tangent, and produce it, making TR equal to ST. Join SP and RP, and produce



the latter to Q. Then SP, PQ are respectively the incident and reflected rays at P. Also $SP = PR$, and therefore the locus of R is the orthotomic surface defined by the equation

$$\rho + \rho' = 0.$$

The locus of R is a curve similar to the pedal of the reflecting curve of twice the linear dimensions. The evolute of this curve is the caustic required.

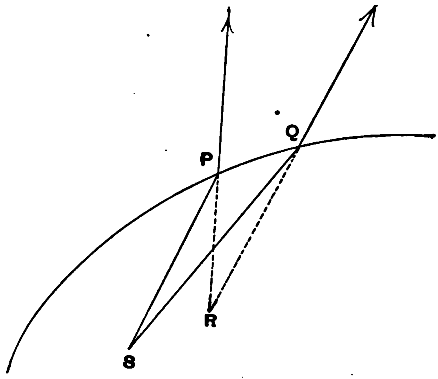


In the case of a circle the equation to the locus of R may be expressed in the form $r = 2\ (a - c \cos\theta)$, where a is the radius of the circle, and c the distance of the bright point from the centre of the circle; so that *the caustic by reflexion, of a circle, is the evolute of a limaçon.*

105. Similarly, to find the caustic by refraction at a curve, a convenient orthotomic curve to construct, is defined by the equation

$$\mu\rho + \mu'\rho' = 0.$$

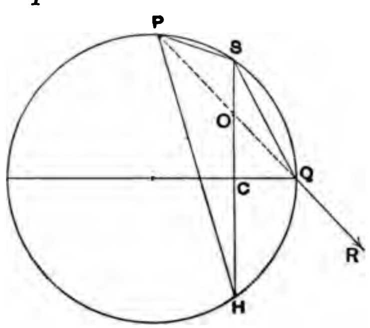
It is easy to see that this curve may be constructed in the manner following. With any point P of the refracting curve as centre, describe a circle of radius ρ' such that $\mu'\rho' = \mu\rho$, where



$\rho = SP$; then the envelope of these circles for different positions of the point P is the orthotomic curve required.

We add geometrical investigations of the caustics by refraction at a line and circle.

106. *To find the caustic by refraction at a straight line, for rays issuing from a point.*



Let S be the bright point; draw SC perpendicular to the line, and produce it to H, so that $CH = CS$. Let SQ be any ray incident at Q, and QR the corresponding refracted ray. Describe a circle about the triangle SHQ, cutting QR in P; then PQ bisects the angle SPH. Let ϕ be the angle of incidence and ϕ' the angle of refraction at Q; then the angle $POS = \phi'$, and

$$\phi = \angle HSQ = \angle HPQ = \angle SPO.$$

Hence $$SO : SP = \sin\phi : \sin\phi',$$

and therefore $$\mu SO = \mu' SP.$$

But since the angle P is bisected,

$$HO : HP = SO : SP$$

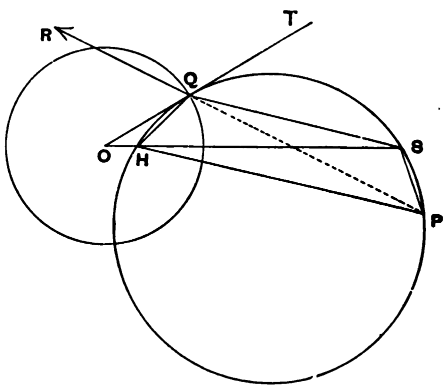
and therefore $$\mu HP = \mu' SP.$$

By addition, $$\mu . SH = \mu' (SP + HP).$$

Thus the locus of P is an ellipse whose foci are S and H; and PQ is normal to the ellipse, and therefore the ellipse is an orthotomic curve. The evolute of this ellipse is the caustic required.

If the second medium is more highly refractive than the first, it may be shown in the same way that the caustic is the evolute of a hyperbola whose foci are S and H.

107. *To find the caustic by refraction at a circle for rays issuing from a point.*



Let O be the centre of the refracting circle, S the origin of light, and SQ an incident ray, QR the corresponding refracted ray. Describe a circle through S and touching the radius OQ in Q; let this circle meet OS in H, and the refracted ray in P. Then $OH . OS = OQ^2$, and therefore H is a fixed point. Also, from the similar triangles OQS, OHQ, $QS : HQ = OS : OQ$, which is a constant ratio. Let ϕ be the angle of incidence, ϕ' that of refraction. Then if OQ be produced to T, $\phi = \angle SQT = \angle QPS$, in the alternate segment; and similarly $\phi' = \angle PQT = \angle QHP =$ supplement of QSP. Then from the triangle QSP,

$$QS : QP = \sin\angle QPS : \sin\angle QSP = \sin\phi : \sin\phi',$$

which is a constant ratio, and therefore also $QH : QP$ is a constant ratio.

Now by Euclid VI. D,

$$QH \,.\, SP + QS \,.\, PH = SH \,.\, QP;$$

and therefore if $SP = \rho,\ PH = \rho'$,

$$\frac{QH}{QP}\rho + \frac{QS}{QP}\rho = SH$$

or $$m\rho + m'\rho' = c, \text{ say.}$$

The locus of P is therefore a Cartesian oval, of which S and H are foci. Also since PQ divides the angle between the radii vectores into two parts whose sines are in the ratio of the chords QS, QH, that is in the ratio $m' : m$, PQ is the normal to the curve. Hence *the caustic is the evolute of a Cartesian oval of which S and H are foci.*

108. This construction fails when the rays are parallel. The equation to the caustic in this case may however be found by a different analysis.

Let ϕ, ϕ' be the angles of incidence and refraction of any ray parallel to the axis of x, so that $\sin\phi' = k\sin\phi$, where $k = \mu/\mu'$. Then if we take the centre of the circle as origin, the equation to the refracted ray is

$$y - a\sin\phi = \tan(\phi - \phi')(x - a\cos\phi),$$

or $$y\cos(\phi - \phi') - x\sin(\phi - \phi') = a\sin\phi'.$$

Now let $\phi = \operatorname{am} u$, then

$$\sin\phi' = k\operatorname{sn} u,\ \cos\phi' = \sqrt{1 - k^2\operatorname{sn}^2 u} = \operatorname{dn} u.$$

Hence the equation to the refracted ray is

$$y(\operatorname{cn} u \operatorname{dn} u + k\operatorname{sn}^2 u) - x(\operatorname{dn} u - k\operatorname{cn} u)\operatorname{sn} u = ak\operatorname{sn} u.$$

Multiply both sides of this equation by $k\operatorname{cn} u + \operatorname{dn} u$; it becomes at once

$$y(\operatorname{cn} u + k\operatorname{dn} u) - xk'^2\operatorname{sn} u = ak\operatorname{sn} u(k\operatorname{cn} u + \operatorname{dn} u)$$

or, dividing by $\operatorname{sn} u$,

$$y\left(\frac{\operatorname{cn} u + k\operatorname{dn} u}{\operatorname{sn} u}\right) - xk'^2 = ak(k\operatorname{cn} u + \operatorname{dn} u).$$

To find the envelope of this line, differentiate with regard to u; we get

$$-y\,\frac{\operatorname{dn} u + k \operatorname{cn} u}{\operatorname{sn} u} = -ak^2 \operatorname{sn} u\,(\operatorname{dn} u + k\operatorname{cn} u),$$

which gives $$y = ak^2 \operatorname{sn}^3 u.$$

Substitute this value of y in the equation to the line; then

$$\begin{aligned} -k'^2 x &= ak\,(k\operatorname{cn} u + \operatorname{dn} u) - ak^2 \operatorname{sn}^2 u\,(\operatorname{cn} u + k\operatorname{dn} u) \\ &= a\,(k^2 \operatorname{cn}^3 u + k\operatorname{dn}^3 u). \end{aligned}$$

We can now easily eliminate u between these equations; and the equation to the caustic becomes

$$-k'^2 x = ak^2\left\{1 - \left(\frac{y}{ak^2}\right)^{\frac{2}{3}}\right\}^{\frac{3}{2}} + ak\left\{1 - \left(\frac{ky}{a}\right)^{\frac{2}{3}}\right\}^{\frac{3}{2}}$$

or, substituting for k and k' their values,

$$(\mu^2 - \mu'^2)\,x = (\mu^{\frac{4}{3}} a^{\frac{2}{3}} - \mu'^{\frac{4}{3}} y^{\frac{2}{3}})^{\frac{3}{2}} + \mu\,(\mu'^{\frac{2}{3}} a^{\frac{2}{3}} - \mu^{\frac{2}{3}} y^{\frac{2}{3}})^{\frac{3}{2}}.$$

This equation is due to St. Laurent, and the method of obtaining it to Mr Glaisher.

If in the equation we interchange μ and μ', and write $\mu a/\mu'$, instead of a, it becomes

$$(\mu'^2 - \mu^2)\,x = \mu\,(\mu'^{\frac{2}{3}} a^{\frac{2}{3}} - \mu^{\frac{2}{3}} y^{\frac{2}{3}})^{\frac{3}{2}} + (\mu^{\frac{4}{3}} a^{\frac{2}{3}} - \mu'^{\frac{4}{3}} y^{\frac{2}{3}})^{\frac{3}{2}},$$

which is the same curve as before; for when we rationalise the equation, the difference of sign on the left will disappear.

Thus *the caustic by refraction for parallel rays of a circle of radius a and index of refraction* μ'/μ, *is the same as that for a concentric circle of radius* $a\mu/\mu'$, *and index of refraction* μ/μ'.

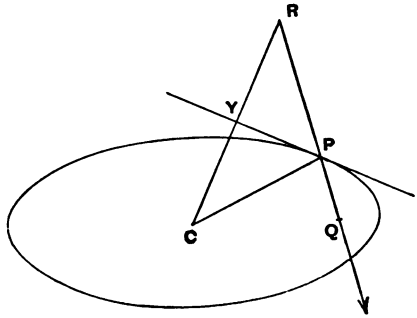
For further information on caustics we refer to Prof. Cayley's "Memoir on Caustics," *Phil. Trans.* 1856.

109. *Caustic by reflexion of an ellipse, the bright point being in the centre.*

Let RPQ be a reflected ray, Q the point where it touches the caustic, R the corresponding point of the secondary caustic. Then RQ is the radius of curvature of the secondary caustic, that is, $RQ = 2\rho$, where ρ is the radius of curvature of the locus of Y, the foot of the perpendicular from C on the tangent. Now if ϖ be the length of the perpendicular on the tangent at Y to the locus of Y, $\varpi r = p^2$, and therefore, differentiating,

$$\frac{d\varpi}{dp}r+\varpi\frac{dr}{dp}=2p.$$

But $$\rho=\frac{pdp}{d\varpi}; \text{ and } \frac{rdr}{dp}=\frac{a^2b^2}{p^3},$$



being the radius of curvature of the ellipse; hence,

$$\frac{r}{\rho}+\frac{p^2}{r}\frac{a^2b^2}{p^3.pr}=2$$

or, $$\frac{a^2b^2}{p^2r^2}=2-\frac{r}{\rho}.$$

Let (u, v) be the coordinates of R; then it may easily be seen that

$$u=\frac{2ab^2\cos\phi}{a^2\sin^2\phi+b^2\cos^2\phi},$$

and $$v=\frac{2ba^2\sin\phi}{a^2\sin^2\phi+b^2\cos^2\phi},$$

where ϕ is the eccentric angle of P.

Also, if (x, y) be coordinates of Q,

$$\frac{x-u}{x-a\cos\phi}=\frac{y-u}{y-b\sin\phi}=\frac{QR}{QP}=\frac{2\rho}{2\rho-r}.$$

Hence, $$\frac{x-u}{x-a\cos\phi}=\frac{2p^2r^2}{a^2b^2},$$

$$=\frac{2r^2}{a^2+b^2-r^2},$$

since $p^2(a^2+b^2-r^2)=a^2b^2$, from the ellipse.

Clearing of fractions, we get

$$x(a^2+b^2-3r^2)=u(a^2+b^2-r^2)-2r^2a\cos\phi.$$

But $$u(a^2+b^2-r^2)=u(a^2\sin^2\phi+b^2\cos^2\phi)$$
$$=2ab^2\cos\phi,$$

and therefore $x(a^2+b^2-3r^2)=2a\cos\phi(b^2-r^2)=-2a\cos^3\phi(a^2-b^2)$.

Similarly $\quad y(a^2+b^2-3r^2)=2a\sin\phi\,(a^2-r^2)=2b\sin^3\phi\,(a^2-b^2).$

Hence, by division, $\qquad \tan\phi = -\dfrac{\left(\dfrac{y}{b}\right)^{\frac{1}{3}}}{\left(\dfrac{x}{a}\right)^{\frac{1}{3}}}.$

Again, eliminating ϕ between these two equations, we get

$$\left\{\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}\right\}(a^2+b^2-3r^2)^{\frac{2}{3}}=\{2(a^2-b^2)\}^{\frac{2}{3}},$$

which becomes $\quad \left\{\left(\dfrac{x}{a}\right)^{\frac{2}{3}}+\left(\dfrac{b}{y}\right)^{\frac{2}{3}}\right\}^{\frac{3}{2}}(a^2+b^2-3r^2)=2(a^2-b^2).$

But $r^2=a^2\cos^2\phi+b^2\sin^2\phi=\dfrac{a^2\left(\dfrac{x}{a}\right)^{\frac{2}{3}}+b^2\left(\dfrac{y}{b}\right)^{\frac{2}{3}}}{\left(\dfrac{x}{a}\right)^{\frac{2}{3}}+\left(\dfrac{y}{b}\right)^{\frac{2}{3}}}.$

Hence, finally,

$$\left\{\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}\right\}^{\frac{1}{2}}\left[(a^2+b^2)\left\{\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}\right\}-3\left\{a^2\left(\frac{x}{a}\right)^{\frac{2}{3}}+b^2\left(\frac{y}{b}\right)^{\frac{2}{3}}\right\}\right]=2(a^2-b^2),$$

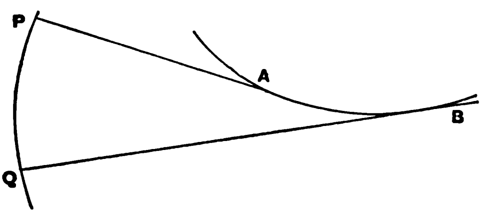
that is, $\quad \left\{\left(\dfrac{x}{a}\right)^{\frac{2}{3}}+\left(\dfrac{y}{b}\right)^{\frac{2}{3}}\right\}^{\frac{1}{2}}\left[\left(\dfrac{x}{a}\right)^{\frac{2}{3}}(\tfrac{1}{2}b^2-a^2)+\left(\dfrac{y}{b}\right)^{\frac{2}{3}}(\tfrac{1}{2}a^2-b^2)\right]=a^2-b^2,$

which is the caustic required.

110. *To find the length of the arc of a caustic.*

The length of the arc of a caustic of any orthotomic system of rays in one plane can always be found. For the caustic is the evolute of the orthogonal curves.

Suppose a system of rays issuing from a point, or normal to a given surface, to be reflected and refracted any number of times.



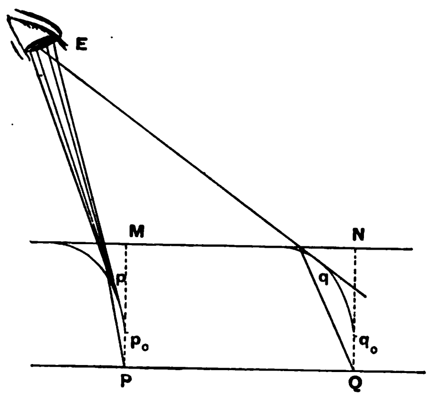
For each ray, form the function $\Sigma\mu\rho$, and let $V=\Sigma\mu\rho$. Let the final medium be of refractive index μ, and let $V=V_0$ be the equation to an orthogonal curve in this medium, say the curve PQ.

Let AB be any arc of the caustic, and let PA, QB be the rays touching at A, B. Then the arc $AB = QB - PA$, by the properties of evolutes.

Also
$$V_A = V_0 + \mu PA,$$
$$V_B = V_0 + \mu QB;$$
and therefore by subtraction,
$$V_B - V_A = \mu\,(\text{arc } AB).$$

111. We can now, by means of caustics, indicate more accurately the manner and position in which an object under water is seen by an eye outside.

Suppose for instance that the water had a horizontal level bottom not very deep. Let P be a point on the bottom, let us trace the pencil of rays by which an eye sees the point P. Draw the normal PM and consider rays in the plane EPM. Construct the caustic in this plane which is touched by refracted rays originally diverging from P. We must draw the two extreme tangents to this



caustic which will meet the eye, and then these will bound the part of the pencil which traverses the air; if we join the points where these tangents meet the surface to P, the joining lines will bound the pencil as it passes through the water. The two tangents to the caustic meet at the point of contact of either of them, very nearly. Thus to an eye outside the point P appears to be at p.

Curves of illumination.

112. If light be incident on a series of bright curves or grooves drawn very close together, so that the reflected light may be received by the eye of a spectator, he will see one or more curves

of special illumination drawn across the grooves. This is very commonly seen when bright rods, such as the spokes of a wheel of a bicycle are revolving in the light of the sun. We shall now consider how these bright curves are produced, and how their forms may be investigated when the reflecting curves are given.

Every point of a reflecting curve will scatter light, and will in this way make itself visible to the eye; but there will be one or more points on the curve which will reflect light to the eye according to the regular laws of reflexion. More light will reach the eye from such a point of the curve than from any other, and the point will therefore appear brighter than the rest of the curve. The locus of these bright points will be a bright curve, whose form is required.

Let the system of reflecting curves be represented by the equation $\phi(x, y) = \alpha$, where α is an arbitrary parameter; and let the incident light proceed from a given luminous point Q. Let E be the position of the eye of the spectator, and P any point on one of the reflecting curves. Draw the tangent to the curve in its own plane. Then, if the reflecting curve be a small groove or a thin rod, an infinite number of tangent planes can be drawn to the groove or rod, all passing through this tangent line. If one of these can be drawn so as to reflect a ray of light proceeding from Q in the direction of PE, then P will be a bright point. In order that this may be possible, we must be able to draw a normal to the groove at P, which shall lie in the plane QPE and bisect the angle QPE. If this can be done the rays QP, PE will make equal angles with the tangent line at P; and, conversely, if this condition be satisfied it is easy to see, as in § 12, that the other two conditions are also satisfied.

Let $(x, y, 0)$ be the coordinates of the point P, (f, g, h) those of the point Q, and (a, b, c) those of the point E. Then if l, m be the direction cosines of the tangent at P,

$$l\phi_x + m\phi_y = 0;$$

and expressing the condition that the lines QP, EP make equal angles with the tangent at P, on opposite sides of it, we get the equation

$$\frac{(f-x)\,l + (g-y)\,m}{\sqrt{(f-x)^2 + (g-y)^2 + h^2}} + \frac{(a-x)\,l + (b-y)\,m}{\sqrt{(a-x)^2 + (b-y)^2 + c^2}} = 0.$$

The ratio $l:m$ may be eliminated by means of the previous equation, and we get the equation

$$\frac{(f-x)\,\phi_y-(g-y)\,\phi_x}{\sqrt{(f-x)^2+(g-y)^2+h^2}}+\frac{(a-x)\,\phi_y-(b-y)\,\phi_x}{\sqrt{(a-x)^2+(b-y)^2+c^2}}=0,$$

and this is the equation to the bright curve required.

113. The same equations may be arrived at in a shorter manner by means of the theorem that the whole length of the path of light from one point to another point is a minimum.

For if P be the bright point, it follows that the length of the path $QP+PE$ must be a minimum, subject to the condition that P shall always lie on the curve $\phi(x, y)=a$. If $(x, y, 0)$, (f, g, h), (a, b, c) be respectively the coordinates of P, Q and E,

$$QP+PE=\sqrt{(x-f)^2+(y-g)^2+h^2}+\sqrt{(x-a)^2+(y-b)^2+c^2}.$$

To make this a minimum, subject to the condition just expressed, we equate to zero the first differential of each equation; we therefore get the equations

$$\phi_x\,dx+\phi_y\,dy=0,$$

$$\frac{(x-f)\,dx+(y-g)\,dy}{\sqrt{(x-f)^2+(y-g)^2+h^2}}+\frac{(x-a)\,dx+(y-b)\,dy}{\sqrt{(x-a)^2+(y-b)^2+c^2}}=0.$$

Eliminating the ratio $dx:dy$ between these equations, we obtain the equation to the bright curve, in the form

$$\frac{(x-f)\,\phi_y-(y-g)\,\phi_x}{\sqrt{(x-f)^2+(y-g)^2+h^2}}+\frac{(x-a)\,\phi_y-(y-b)\,\phi_x}{\sqrt{(x-a)^2+(y-b)^2+c^2}}=0.$$

114. As an example of the foregoing process, let us find the form of the bright curves seen on the spokes of a bicycle wheel revolving in the sun-light.

Take the axis of the wheel as the axis of z, and suppose the direction of the sun's rays to be defined by the direction-cosines (l, m, n), and let the coordinates of the eye be (a, b, c). Then supposing the wheel to be plane, the equation to the reflecting curves will be of the form $y=x\tan\theta$; and therefore, if we express the fact that the incident and reflected rays make equal angles

with the line whose direction-cosines are $(\cos\theta, \sin\theta, 0)$, on opposite sides of it, we find

$$l\cos\theta + m\sin\theta + \frac{(a-x)\cos\theta + (b-y)\sin\theta}{\sqrt{(a-x)^2+(b-y)^2+c^2}} = 0.$$

Eliminating θ, the equation to the bright curves becomes

$$(lx+my)^2\{(a-x)^2+(b-y)^2+c^2\} = \{(a-x)x+(b-y)y\}^2.$$

EXAMPLES.

1. A luminous point is placed at a distance h in front of a plane refracting surface. Show that the orthotomic surfaces of the rays within the medium are those formed by the revolution of the curves

$$\frac{\mu x}{\sqrt{a^2-\mu^2y^2}} + \frac{h}{\sqrt{a^2-y^2}} = 1$$

about the axis of x, a being a variable parameter, the origin being the foot of the perpendicular from the luminous point on the plane boundary, and the axis of x normal to that boundary.

2. Rays emanating from the focus of a parabola are reflected from the evolute of the parabola, show that the caustic is the evolute of a parabola.

3. If rays emanating from the vertex are reflected from a parabola, the caustic is the evolute of a cissoid.

4. When the luminous point is the centre, the caustic by reflexion of the involute of a circle is the evolute of the spiral of Archimedes.

5. Rays emanate from the pole of a plane curve whose equation is given in the form

$$f(r, p) = 0 \qquad\qquad (1),$$

show that the equation to the katacaustic will be the result of eliminating r and p between (1) and the equations

$$\sqrt{r'^2-p'^2} + \sqrt{r^2-p'^2} = \frac{rp\dfrac{dr}{dp}}{2r - p\dfrac{dr}{dp}} \qquad\qquad (2),$$

$$p'^2 = \frac{4p^2(r^2-p^2)}{r^2} \qquad\qquad (3).$$

Conversely, if the equation to the katacaustic be

$$\phi(r', p') = 0 \qquad\qquad (4),$$

the equation of the reflecting curve will be obtained by eliminating r', p' between the equations (2), (3), (4).

If the reflecting curve be the involute of a circle, represented by the equation $r^2=p^2+a^2$, the equation to the caustic will be

$$r^2(8a^2-p^2)^2=4a^2\{16a^4+p^2(4a^2-p^2)\}.$$

If the reflecting curve be the hyperbolic spiral, $r\theta=a$ or $(r^2+a^2)p^2=a^2r^2$, the equation to the caustic will be

$$r(a-p)=ap.$$

If the caustic be a circle $r=a$, show that the reflecting curve is determined by the equation

$$a^2r^2=4p^2(r^2-p^2).$$

6. A luminous point is placed in front of a thick plate of glass with parallel faces; show that the caustics produced by the successive reflexions and refractions at the surfaces of the plate are the evolutes of two series of equal and similarly placed prolate quadrics of revolution, each of which has at least one focus coincident with one of the successive reflexion-images of the point due to the faces of the plate considered as plane mirrors situated in air.

7. If rays from a luminous point be reflected at a parabola, show that the katacaustic has three asymptotes, reflected from points at finite distance, except when the luminous point is on the axis, when there are only two, and that then the abscissa of the point where the asymptote is reflected is one-third the abscissa of the luminous point.

8. Prove that, if rays of light proceed from a point and be reflected at a conic whose plane contains the radiant point, the reflected rays are all normal to a bicircular quartic which has the radiant point as double point. If the radiant point be the centre of the conic, show that the equation of the quartic may be written

$$r^2=A\cos 2\theta+B.$$

9. A luminous point moves along a diameter of a reflecting circle, of radius a; prove that the two cusps of the caustic, which are not on that diameter, move on the curve $r=a\cos\frac{1}{2}\theta$.

10. Rays proceeding from a luminous point in the pole of the spiral $r=ae^{\theta\cot\alpha}$ are reflected at the curve; show that the caustic is a similar spiral. Also show that the spiral will be its own caustic if

$$\tfrac{1}{2}\sec\alpha=e^{(2n\pi-\alpha)\cot\alpha}$$

where n is any positive integer.

11. Rays parallel to the axis of y are incident on the reflecting curve $y=e^x$, show that the equation to the caustic is the catenary

$$y=\tfrac{1}{2}\{e^{x+1}+e^{-(x+1)}\}.$$

12. A ray proceeding from a point in the circumference of a circle is reflected n times at the circle; prove that the point of intersection with the consecutive ray similarly reflected is at a distance from the centre equal to $a/(2n+1)\sqrt{1+4n(n+1)\sin^2\theta}$ where a is the radius of the circle, and θ the angle of incidence of the ray. Prove also that the caustic surface generated by such rays is the surface of revolution generated by an epicycloid in which the fixed circle has the radius $a/(2n+1)$, and the moving circle the radius $na/(2n+1)$.

13. Prove that the caustic for a pencil of parallel rays refracted at a circle of radius unity and refraction-coefficient μ, is given by the equations

$$\mu^2(1-\mu^2)\,x=\mu^2\cos^3\phi+(\mu^2-\sin^2\phi)^{\frac{3}{2}}\,;\ \mu^2 y=\sin^3\phi.$$

14. Light emanating from a point O is reflected at a curve so that the caustic is a circle whose centre is O and radius equal to a. Prove that the curve must belong to one of the families

$$\theta+\text{const.}=\frac{r}{a}\pm\left\{\frac{\sqrt{r^2-a^2}}{a}+\sin^{-1}\frac{a}{r}\right\}.$$

15. Rays issuing from the centre of a given circle are refracted at a curve so that the refracted rays are all tangents to the circle. Find the equation to the refracting curve.

16. Rectify the caustic in the case of rays parallel to the axis of x falling on the reflecting curve

$$\sin(ay)=e^{ax}.$$

17. At a point on the inside of a polished hollow right circular cylinder of radius a is placed a luminous point; explain the formation of a series of bright curves on any plane at right angles to the axis of the cylinder; and prove that they are all epicycloids, the radius of the rolling circle for the nth curve being $na/(2n+1)$, and that of the fixed circle $a/(2n+1)$.

18. The surface of a hollow right cone is grooved with an infinite number of circular grooves. A bright point is placed on the surface; prove that an eye situated on the opposite generating line will see a bright curve which lies on a sphere of radius $abc/(a^2-b^2)$, passing through the vertex of the cone, where a, b are the distances from the vertex of the bright point and the eye, and c is the distance between them.

19. If a series of fine smooth grooves be cut in a plane surface in the shape of concentric circles, the bright curve formed by reflection of the light from a luminous point, and seen by an eye situate in the plane through the luminous point and the axis of the circles, will be a circle.

20. Fine polished wire with circular transverse section is disposed along the meridians of a sphere whose axis is directed to the sun. Prove that those reflected rays which have a common direction normal to the axis proceed from curves in which the sphere is met by an elliptic cone, the planes of whose circular sections are inclined at half a right angle.

21. In a hollow ellipsoidal shell small polished grooves are made coinciding with one series of circular sections, and a bright point is placed at one of the umbilics in which the series terminates; prove that the locus of the bright points seen by an eye in the opposite umbilic is a central section of the ellipsoid, and that the whole length of the path of any ray by which a bright point is seen is constant.

22. A bicycle wheel in which the spokes are perpendicular to the axis is placed in the sun and spun rapidly. Show that the equation of the bright curve seen on the spokes by an eye in the axis of the wheel produced is of the form

$$r^2(\sec^2\theta\sec^2\alpha-1)=a^2,$$

α denoting the angle between the direction of the sun's rays and the plane of the wheel, and a the distance of the eye from the wheel.

23. A man standing on the sea-shore sees the light of a star reflected on the surface of the sea when it is covered with gentle ripples travelling in all directions, find the equation to the boundary of the bright patch on the water, considering the undisturbed surface of the sea to be a horizontal plane.

Find the condition that this patch should reach to infinity.

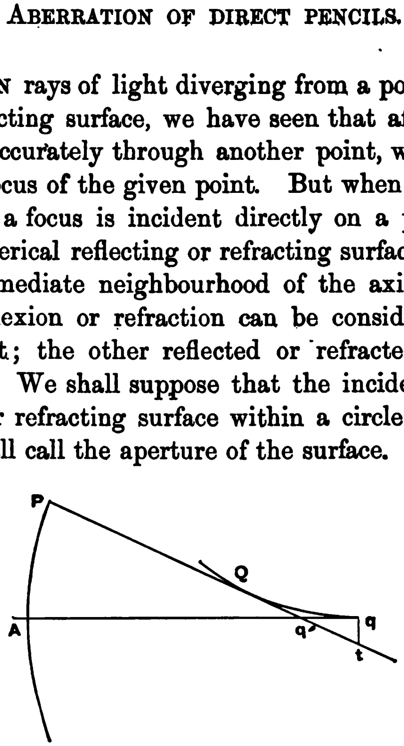
If z be the zenith distance of the star, and the tangents to the boundary of the bright patch from the man's feet contain an angle 4θ, show that if the patch do not extend to infinity, the angle which it subtends at the man's eye in a vertical plane passing through the star is

$$4\tan^{-1}(\sin\theta\tan z).$$

CHAPTER VII.

ABERRATION OF DIRECT PENCILS.

115. WHEN rays of light diverging from a point are incident on a plane reflecting surface, we have seen that after reflexion all the rays pass accurately through another point, which was called the conjugate focus of the given point. But when a pencil of rays diverging from a focus is incident directly on a plane refracting surface or a spherical reflecting or refracting surface, it is only the rays in the immediate neighbourhood of the axis of the pencil which after reflexion or refraction can be considered as passing through a point; the other reflected or refracted rays touch a caustic surface. We shall suppose that the incident pencil meets the reflecting or refracting surface within a circle of small radius y, which we shall call the aperture of the surface.



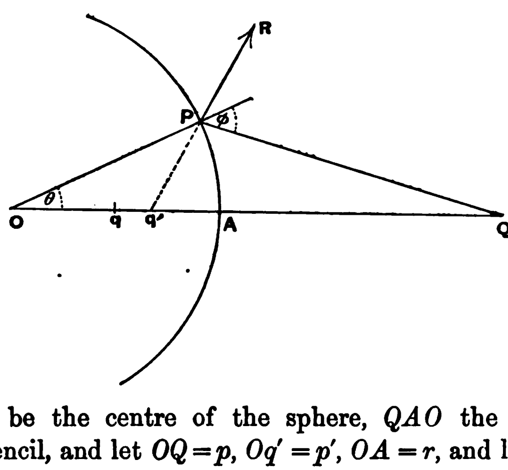
Let q be the focus of the rays which are in the immediate neighbourhood of the axis, and let Pq' be the extreme ray after reflexion or refraction, cutting the axis in q', and a plane through q perpendicular to the axis in t. Then qq' is called the *longitudinal aberration* of the ray Pq', and qt its *lateral aberration.* These

aberrations may be expressed approximately in terms of the aperture when the aperture is small.

We need only find the aberration for pencils which diverge from points on the axis. For if a pencil diverge from a point not on the axis, the image will lie on the line joining the origin of light to the centre of the reflecting or refracting surface; and this line will be the axis of the pencil and the longitudinal aberration along this line will be known. This must be projected on the original axis by multiplying by the cosine of the inclination of the line to that axis. This inclination will be very small, so that when multiplied by the small aberration its cosine may be taken to be unity; and therefore to our present approximation, the longitudinal aberration is the same for all points lying in a plane perpendicular to the axis.

116. *To find the aberration of a pencil directly reflected at a spherical surface.*

Let QPR be the path of the extreme ray, and let PR produced backwards meet the axis in q'.



Let O be the centre of the sphere, QAO the axis of the incident pencil, and let $OQ=p$, $Oq'=p'$, $OA=r$, and let the angle POA be denoted by θ, and the angle of incidence of the ray QP, by ϕ. Then

$$\frac{r}{p}=\frac{\sin(\phi-\theta)}{\sin\phi},$$

$$\frac{r}{p'}=\frac{\sin(\phi+\theta)}{\sin\phi};$$

whence $$\frac{r}{p}+\frac{r}{p'}=\frac{\sin(\phi-\theta)+\sin(\phi+\theta)}{\sin\phi}=2\cos\theta,$$

or $$\frac{1}{p}+\frac{1}{p'}=\frac{2\cos\theta}{r}.$$

But if θ were made indefinitely small, we should have

$$\frac{1}{p}+\frac{1}{p_0'}=\frac{2}{r},$$

and therefore $$\frac{1}{p_0'}-\frac{1}{p'}=\frac{2}{r}(1-\cos\theta).$$

In this equation, powers of θ above the second may be neglected, and p' is very nearly equal to p_0'; the equation may therefore be written

$$\frac{p'-p_0'}{p'^2}=\frac{\theta^2}{r};$$

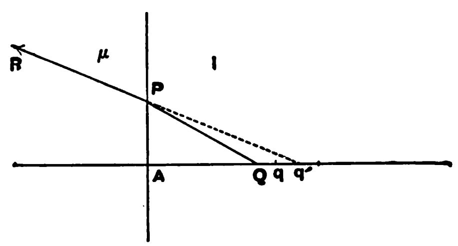
and therefore, since $y=r\sin\theta=r\theta$, approximately, we get

$$qq'=\frac{y^2}{r^3}p'^2.$$

This is the value of the longitudinal aberration of the extreme ray. We notice that qq' has the same sign as r, and therefore *if we stand at the mirror and look towards the centre, the caustic points in this direction in all cases.*

When the rays are incident parallel to the axis of the mirror, q will be at the principal focus of the mirror, so that Oq will be f, where f is the focal length of the mirror; in this case the longitudinal aberration will be $$qq'=-\frac{y^2}{4f}.$$

117. *To find the aberration of a pencil directly refracted at a plane surface.*



Let QPR be the course of the extreme ray, so that AP is the radius of the aperture. Let PR produced backwards meet the

axis in q', and let $AQ = u$, $Aq' = u'$, $AP = y$. Then if ϕ, ϕ' be the angles of incidence and refraction, $\sin\phi = AP/PQ$, $\sin\phi' = AP/Pq'$, and therefore $Pq' = \mu PQ$. Expressing these distances in terms of u, u', y, we have

$$u'^2 + y^2 = \mu^2 (u^2 + y^2),$$

or
$$u' = \{\mu^2 u^2 + (\mu^2 - 1)\, y^2\}^{\frac{1}{2}}.$$

When the aperture is small, we may neglect all powers of y beyond the second, and we get, approximately,

$$u' = \mu u + \tfrac{1}{2}(\mu^2 - 1)\frac{y^2}{\mu u}.$$

But if we make $y = 0$, we get the value of Aq, namely,

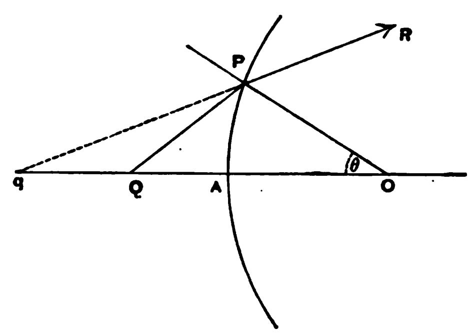
$$Aq = \mu u.$$

Hence, by subtraction, $$qq' = \tfrac{1}{2}(\mu^2 - 1)\frac{y^2}{\mu u},$$

and this is the longitudinal aberration of the extreme ray.

118. *To find the aberration of a pencil directly refracted at a spherical surface.*

Let QPR be the path of any ray of the pencil whose axis is QAO, and let PR produced backwards meet the axis in q. Let $OQ = p$, $Oq = q$, and $AO = r$.



The formula for the aberration of the extreme ray will be much simplified if we express it in terms of the reciprocals of the quantities p, q, r; let u, v, ρ denote the reciprocals of the quantities p, q, r, respectively. Let the angle POA be denoted by

θ, and the angles of incidence and refraction by ϕ and ϕ', respectively. Then

$$\frac{\sin\phi}{\sin\theta}=\frac{QO}{QP}, \quad \frac{\sin\phi'}{\sin\theta}=\frac{qO}{qP};$$

and therefore, by the law of refraction,

$$\frac{QO}{QP}=\mu\,\frac{qO}{qP}.$$

This relation, when expressed in terms of p, q, r becomes

$$\frac{p^2}{p^2+r^2-2pr\cos\theta}=\mu^2\,\frac{q^2}{q^2+r^2-2qr\cos\theta},$$

or, in reciprocals,

$$v^2+\rho^2-2v\rho\cos\theta=\mu^2(u^2+\rho^2-2u\rho\cos\theta)\ldots\ldots(1),$$

an equation which gives the relation between v and u for all values of θ.

119. There is one case in which the value of v is independent of θ; this is when $v=\mu^2u$, so that the coefficient of $\cos\theta$ vanishes. If we substitute this value of v in the previous equation, we get $\mu^4u^2+\rho^2=\mu^2u^2+\mu^2\rho^2$, which gives $\mu u=\rho$, or finally, $p=\mu r$. This result is expressed in the following theorem. *If a pencil of rays diverge from a point whose distance from the centre of the refracting surface is μ times the radius, the refracted rays will all pass accurately through another point, whatever be their incidence.*

If in equation (1) we make $\theta=0$, we get

$$v_0-\rho=\mu(u-\rho)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (2),$$

which is equivalent to the relation between the abscissæ of conjugate points already found when the rays all lie in the immediate neighbourhood of the axis.

To find a more closely approximate value of v, we may suppose the aperture so small that powers of θ beyond the second may be neglected. Then the equation (1) may be written

$$(v-\rho)^2+v\rho\theta^2=\mu^2\{(u-\rho)^2+u\rho\theta^2\},$$

or, extracting the square root of both sides by the Binomial Theorem,

$$(v-\rho)\left\{1+\frac{v\rho\theta^2}{2(v-\rho)^2}\right\}=\mu(u-\rho)\left\{1+\frac{u\rho\theta^2}{2(u-\rho)^2}\right\}.$$

By means of equation (2) this reduces to

$$v - v_0 = \tfrac{1}{2}\rho\theta^2\left\{\frac{\mu u}{u-\rho} - \frac{v}{v-\rho}\right\} \quad\text{.................. (3)},$$

where on the right we may suppose v to have its first approximate value v_0.

This formula contains the whole theory of aberration at a spherical surface.

120. It is generally more convenient to measure the distances from the surface, and not from the centre. The formula (3) may easily be transformed in terms of the new variables. Let α, β denote the reciprocals of AQ, Aq, measured from the surface to the right; then

$$\frac{1}{\alpha} = \frac{1}{\rho} - \frac{1}{u},$$

and

$$\frac{1}{\beta} = \frac{1}{\rho} - \frac{1}{v}.$$

The equation (2) now becomes

$$\alpha - \rho = \mu(\beta - \rho) \quad\text{........................ (4)},$$

and the equation (3),

$$dv = \tfrac{1}{2}\theta^2\{\mu\alpha - \beta\}.$$

But by differentiation, $\dfrac{d\beta}{\beta^2} = -\dfrac{dv}{v^2}$,

and therefore $\qquad dv = \dfrac{v^2}{\beta^2}d\beta = -\dfrac{\rho^2}{(\beta-\rho)^2}d\beta.$

Hence $\qquad d\beta = \tfrac{1}{2}\rho^2\theta^2(\beta-\rho)^2(\beta - \mu\alpha),$

or $\qquad d\beta = \tfrac{1}{2}(\beta-\rho)^2(\beta-\mu\alpha)\,y^2 \quad\text{.................. (5)}.$

If β be eliminated by means of the equation (4), this result becomes

$$d\beta = \frac{(\mu-1)}{2\mu^3}(\rho-\alpha)^2\{\rho - (\mu+1)\alpha\}\,y^2 \quad\text{............ (6)}.$$

121. *To find the aberration in the refraction of direct pencils by lenses.*

Let α, β denote the reciprocals of the distances from the first surface, of the points where the axis is met by the incident and refracted rays, respectively, and let β', α' denote similar quantities

with reference to the second surface, and let θ be the reciprocal of the thickness of the lens, and ρ, ρ' the curvatures of the bounding surfaces of the lens; then, when the aperture is made very small, the relations among these quantities are

$$\alpha - \rho = \mu(\beta - \rho), \qquad \alpha' - \rho' = \mu(\beta' - \rho'), \qquad \frac{1}{\beta} - \frac{1}{\beta'} = \frac{1}{\theta}.$$

Let y, y' be the radii of the apertures of the first and second surfaces respectively; then since α' may be regarded as a function of the two variables β' and y', we must have

$$d\alpha' = \left(\frac{d\alpha'}{d\beta'}\right) d\beta' + \left(\frac{d\alpha'}{dy'}\right) dy',$$

where the differential coefficients are partial.

But if we differentiate the second and third equations, we get

$$\left(\frac{d\alpha'}{d\beta'}\right) = \mu, \qquad \frac{d\beta'}{\beta'^2} = \frac{d\beta}{\beta^2}.$$

Let the variation of β due to the change of aperture at the first surface be denoted by κy^2; and supposing β' fixed, let the variation of α' due to the change of aperture at the second surface be denoted by $\kappa' y'^2$; then $d\beta = \kappa y^2$, and therefore

$$\left(\frac{d\alpha'}{d\beta'}\right) d\beta' = \mu \frac{\beta'^2}{\beta^2} \kappa y^2;$$

and also

$$\left(\frac{d\alpha'}{dy'}\right) dy' = \kappa' y'^2 = \kappa' \frac{\beta^2}{\beta'^2} y^2,$$

since $y' : y = 1/\beta' : 1/\beta$, by similar triangles.

Substituting these values in the expression for $d\alpha'$, we get

$$d\alpha' = y^2 \left\{\mu\kappa \frac{\beta'^2}{\beta^2} + \kappa' \frac{\beta^2}{\beta'^2}\right\}.$$

We have now to substitute the values of κ, κ' as found in the previous investigation. It was there shown that

$$d\beta = \kappa y^2 = \tfrac{1}{2}(\beta - \rho)^2 (\beta - \mu\alpha) y^2;$$

and if we eliminate ρ by means of the equation $\alpha - \rho = \mu(\beta - \rho)$, we find the value of κ to be

$$\kappa = \frac{1}{2(\mu - 1)^2} (\beta - \alpha)^2 (\beta - \mu\alpha).$$

The value of κ' may be found by substituting β' and α' for α and β, respectively, and $1/\mu$ for μ; and therefore

$$\kappa' = -\frac{\mu}{2(\mu-1)^2}(\beta'-\alpha')^2(\beta'-\mu\alpha').$$

Substituting these values of κ, κ' in the expression for $d\alpha'$, it becomes

$$d\alpha' = \frac{\mu y^2}{2(\mu-1)^3}\left\{\frac{\beta'^2}{\beta^2}(\beta-\alpha)^2(\beta-\mu\alpha) - \frac{\beta^2}{\beta'^2}(\beta'-\alpha')^2(\beta'-\mu\alpha')\right\}$$

which is the general expression for $d\alpha'$ for any lens, of whatever thickness it may be. The quantities β, β' may be expressed in terms of α, ρ, α', ρ', and then we get a value of $d\alpha'$, which is a symmetrical function of α, ρ and α', ρ'.

122. When the thickness of the lens may be neglected as inconsiderable, $\beta' = \beta$, and therefore

$$d\alpha' = \frac{\mu y^2}{2(\mu-1)^2}\{(\beta-\alpha)^2(\beta-\mu\alpha) - (\beta-\alpha')^2(\beta-\mu\alpha')\}.$$

If the expression within the brackets be multiplied out, it becomes

$$(\mu+2)(\alpha'-\alpha)\beta^2 - (2\mu+1)(\alpha'^2-\alpha^2)\beta + \mu(\alpha'^3-\alpha^3);$$

we therefore get

$$d\alpha' = \frac{\mu(\alpha'-\alpha)}{2(\mu-1)^2}y^2\{(\mu+2)\beta^2 - (2\mu+1)(\alpha+\alpha')\beta + \mu(\alpha^2+\alpha\alpha'+\alpha'^2)\}.$$

This value of the aberration of a thin lens may be expressed in a more symmetrical form by eliminating β. From the equations

$$\alpha - \rho = \mu(\beta-\rho), \quad \alpha' - \rho' = \mu(\beta-\rho'),$$

we find

$$\beta - \alpha = \frac{\mu-1}{\mu}(\rho-\alpha),$$

$$\beta - \mu\alpha = \frac{\mu-1}{\mu}\{\rho-(\mu+1)\alpha\},$$

with similar expressions for $\beta-\alpha'$, $\beta-\mu\alpha'$ in terms of ρ', α'; substituting these values in the expression for $d\alpha'$, it becomes

$$d\alpha' = \frac{\mu-1}{2\mu^2}y^2[(\rho-\alpha)^2\{\rho-(\mu+1)\alpha\} - (\rho'-\alpha')^2\{\rho'-(\mu+1)\alpha'\}].$$

This symmetrical expression for dx', combined with the equation

$$\alpha' - \alpha = (\mu - 1)\,(\rho - \rho'),$$

contains the whole theory of spherical aberration in a thin lens.

123. When the incident rays are parallel, $\alpha = 0$, and $\alpha' = \phi = (\mu - 1)\,(\rho - \rho')$; the value of dx' then becomes

$$d\phi = \frac{\mu - 1}{2\mu^2}\, y^2 \left[\rho^3 + \{\mu\,(\rho - \rho') - \rho\}^2 \{\mu^2\,(\rho - \rho') - \rho\}\right].$$

If the quantity within the square brackets be developed by multiplication, it becomes

$$\begin{aligned} &\mu\,(\rho - \rho')\,\{\mu^3\,(\rho - \rho')^2 - \mu\,(2\mu + 1)\,(\rho - \rho')\,\rho + (\mu + 2)\,\rho^2\} \\ &= \mu\,(\rho - \rho')\,\{(2 - 2\mu^2 + \mu^3)\,\rho^2 + (\mu + 2\mu^2 - 2\mu^3)\,\rho\rho' + \mu^3\rho'^2\}; \end{aligned}$$

and therefore

$$d\phi = \frac{(\mu - 1)}{2\mu}\,(\rho - \rho')\,y^2\,\{(2 - 2\mu^2 + \mu^3)\,\rho^2 + (\mu + 2\mu^2 - 2\mu^3)\,\rho\rho' + \mu^3\rho'^2\}.$$

124. Several cases may be considered, in order to compare the advantages of lenses of particular forms.

In a plano-spherical lens, having its plane side towards the incident light, $\rho = 0$; and therefore, omitting the dash from the curvature of the second surface,

$$d\phi = -\tfrac{1}{2}\,\mu^2\,(\mu - 1)\,\rho^3 y^2.$$

And in this case $\phi = -(\mu - 1)\,\rho$; therefore

$$d\phi = \tfrac{1}{2}\left(\frac{\mu}{\mu - 1}\right)^2 \phi^3 y^2.$$

In a plano-spherical lens, having its curved side turned towards the incident light, $\rho' = 0$, and

$$d\phi = \frac{\mu - 1}{2\mu}\,(\mu^3 - 2\mu^2 + 2)\,\rho^3 y^2;$$

and since $\phi = (\mu - 1)\,\rho$, this becomes

$$d\phi = \frac{\mu^3 - 2\mu^2 + 2}{2\mu\,(\mu - 1)^2}\,\phi^3 y^2.$$

In a double-convex lens, which has surfaces of equal curvature, $\rho' = -\rho$, and the value of $d\phi$ becomes

$$d\phi = \frac{\mu - 1}{\mu} (4\mu^3 - 4\mu^2 - \mu + 2) \rho^3 y^2.$$

Or, since in this case $\phi = (\mu - 1) 2\rho$,

$$d\phi = \frac{4\mu^3 - 4\mu^2 - \mu + 2}{8\mu (\mu - 1)^2} \phi^3 y^2.$$

When the lens is made of crown glass, in which $\mu = \frac{3}{2}$ nearly, the coefficients of $\phi^3 y^2$ become respectively, $\frac{9}{2}$, $\frac{7}{6}$, $\frac{5}{3}$. Now $\phi = 1/f$, so that $d\phi = -df/f^2$; the values of the aberrations in the three cases considered are therefore $-\frac{9}{2}y^2/f$, $-\frac{7}{6}y^2/f$ and $-\frac{5}{3}y^2/f$, respectively. Thus the best of these forms is that of the plano-spherical lens, with its curved surface turned towards the incident light, and the aberration is greatest in the same lens with its plane surface turned towards the incident light.

We may notice here that in a thin lens of glass, whose semi-diameter is y and focal length is f, the thickness of the lens is equal to y^2/f. For it is easy to see from properties of the circle that the thickness of the lens is equal to $y^2/2r - y^2/2r'$, very nearly, where r, r' are the radii of the curved surfaces. Also

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{r'}\right) = \frac{1}{2}\left(\frac{1}{r} - \frac{1}{r'}\right),$$

and therefore the thickness is equal to y^2/f.

This gives a more definite meaning to the preceding results.

125. We shall next find the form of a lens which will refract a pencil of light issuing from a given point, and bring it to a focus at another given point, with the minimum of aberration.

In this problem α and α' are given, and β is the variable whose value is to make $d\alpha'$ a minimum. We must therefore choose β so that

$$(\mu + 2) \beta^2 - (2\mu + 1)(\alpha + \alpha') \beta + \mu (\alpha^2 + \alpha\alpha' + \alpha'^2)$$

may be a minimum.

This expression may be written in the form

$$\frac{1}{4(\mu + 2)} \left[\{(\mu + 2) 2\beta - (2\mu + 1)(\alpha + \alpha')\}^2 + 4\mu (\mu + 2)(\alpha^2 + \alpha\alpha' + \alpha'^2) - (2\mu + 1)^2 (\alpha + \alpha')^2 \right],$$

and therefore, for a minimum,

$$(\mu + 2)\, 2\beta = (2\mu + 1)(\alpha + \alpha'),$$

and the expression reduces to

$$\frac{1}{\mu + 2}\{(\mu^2 + 2\mu)(\alpha^2 + \alpha\alpha' + \alpha'^2) - (\mu^2 + \mu + \tfrac{1}{4})(\alpha + \alpha')^2\}$$

$$= \frac{1}{\mu + 2}\{(\mu - \tfrac{1}{4})(\alpha' - \alpha)^2 - (\mu - 1)^2 \alpha\alpha'\}.$$

The minimum value of $d\alpha'$ is therefore

$$d\alpha' = \frac{\mu(\alpha' - \alpha)}{2(\mu + 2)} y^2 \left\{\frac{\mu - \frac{1}{4}}{(\mu - 1)^2}(\alpha' - \alpha)^2 - \alpha\alpha'\right\}.$$

To find the form of the lens, we have only to substitute the value of β obtained above, in the equations

$$\left.\begin{aligned}(\mu - 1)\rho &= \mu\beta - \alpha \\ (\mu - 1)\rho' &= \mu\beta - \alpha'\end{aligned}\right\},$$

and we get

$$\left.\begin{aligned}\rho &= p\alpha' + q\alpha \\ \rho' &= p\alpha + q\alpha'\end{aligned}\right\},$$

where

$$p = \frac{2\mu^2 + \mu}{2(\mu - 1)(\mu + 2)}, \quad q = \frac{2\mu^2 - \mu - 4}{2(\mu - 1)(\mu + 2)}.$$

The curvatures of the bounding surfaces are therefore determined.

126. The form of the lens will depend upon the positions of the point from which the light is proceeding, and that at which the rays unite.

When the incident rays are parallel, $\alpha = 0$ and $\alpha' = \phi$, where ϕ denotes the reciprocal of the focal length; and the minimum value of $d\alpha'$ becomes

$$d\phi = \frac{\mu(\mu - \frac{1}{4})}{2(\mu + 2)(\mu - 1)^2}\phi^3 y^2.$$

For a lens of crown glass $\mu = \frac{3}{2}$ nearly, and therefore $d\phi = \frac{15}{14}\phi^3 y^2$; this makes the aberration equal to $-\frac{15}{14}y^2/f$.

The form of the lens is determined, in this case, by the equations $\rho = p\phi$, $\rho' = q\phi$. Accordingly, *the ratio of the curvatures*

of the surfaces is independent of the power of the lens, and is

$$\frac{\rho'}{\rho}=\frac{q}{p}=\frac{2\mu^2-\mu-4}{2\mu^2+\mu}.$$

When $\mu=\frac{3}{2}$, this ratio becomes

$$\frac{\rho'}{\rho}=-\frac{1}{6}.$$

The curvatures of the two surfaces of the lens lie in opposite directions, so that the lens is either double convex or double concave; and the curvature of the posterior surface is $\frac{1}{6}$th of that of the anterior. Such a lens is called a *crossed* lens.

If the index of refraction be such as to satisfy the equation $2\mu^2-\mu-4=0$, or $\mu=\frac{1}{4}(1+\sqrt{33})=1{\cdot}686$ nearly, which is about the value of μ for the more highly refracting kinds of glass, then $\rho'=0$, and the lens will have its posterior surface plane.

For a crossed lens, the aberration for parallel rays is $-\frac{15}{14}y^2/f$; while for a plano-spherical lens whose curved surface is turned towards the incident light, the aberration is $-\frac{9}{8}y^2/f$. The plano-spherical lens is therefore nearly as good as the crossed lens; it is much easier to make, and is therefore much more commonly used.

When plano-spherical lenses are used as objectives for microscopes, the rays diverge from a point very near to the surface of the lens, and emerge nearly parallel to each other, so that it is the plane surface which must be presented to the object.

127. The aberration of any thin lens can be expressed in a simple form in the notation of the last article. For a lens of minimum aberration, it was shown that

$$\beta=\frac{2\mu+1}{2(\mu+2)}(\alpha+\alpha').$$

Assume therefore for *any* lens,

$$\beta=\frac{(2\mu+1)}{2(\mu+2)}(\alpha+\alpha')+\frac{\mu-1}{\mu+2}\epsilon,$$

and for brevity, let

$$\frac{\mu}{\mu+2}=m,\quad \frac{\mu-\frac{1}{4}}{(\mu-1)^2}=n;$$

then if we make these substitutions in the expression for the aberration, it becomes

$$d\alpha' = \tfrac{1}{2}m\,(\alpha' - \alpha)\,y^2\,\{n\,(\alpha' - \alpha)^2 - \alpha\alpha' + \epsilon^2\}.$$

The curvatures of the surfaces may be expressed in terms of α, α', ϵ by means of the equations $(\mu - 1)\rho = \mu\beta - \alpha$, $(\mu - 1)\,\rho' = \mu\beta - \alpha'$, and using p, q, m with the same meanings as before, the values of the curvatures are found to be

$$\left.\begin{aligned} \rho &= p\alpha' + q\alpha + m\epsilon \\ \rho' &= p\alpha + q\alpha' + m\epsilon \end{aligned}\right\}.$$

128. If we wish to make the lens aplanatic, that is, such that the aberration vanishes, the equation of condition is

$$\epsilon^2 = \alpha\alpha' - n\,(\alpha' - \alpha)^2.$$

A primary condition is therefore that α and α' have the same sign; and further, they must be such as to make $\alpha\alpha' > n\,(\alpha' - \alpha)^2$.

These conditions can never be fulfilled for parallel rays; for then $\alpha = 0$, $\alpha' = \phi$, and the value of ϵ^2 is determined by the equation

$$\epsilon^2 = -n\phi^2,$$

which gives an imaginary value of ϵ.

129. *Aberration of a pencil directly refracted through any number of spherical surfaces arranged symmetrically along an axis.*

Let μ, μ', μ''... be the successive refractive indices of the media; let α, β be the reciprocals of the distances of the point and its first image, respectively, from the vertex of the first surface; and let ρ be the curvature of this surface, and let dashed letters denote the corresponding quantities for the other surfaces in succession, all these distances being measured from left to right. Then the following relations hold between these quantities:

$$\left.\begin{aligned} \mu\,(\alpha - \rho) &= \mu'\,(\beta - \rho), \\ \mu'\,(\alpha' - \rho') &= \mu''\,(\beta' - \rho'), \\ \mu''\,(\alpha'' - \rho'') &= \mu'''\,(\beta'' - \rho''), \\ &\cdots\cdots \end{aligned}\right\}\quad\text{.............(1).}$$

If we denote the reciprocals of the thicknesses of the media

between the surfaces, measured along the axis, by $\theta, \theta' \ldots$, we must have also

$$\left.\begin{aligned} \frac{1}{\beta} - \frac{1}{\alpha'} &= \frac{1}{\theta}, \\ \frac{1}{\beta'} - \frac{1}{\alpha''} &= \frac{1}{\theta'}, \\ &\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Let the radii of the apertures of the different surfaces through which the pencil passes be $y, y', y'' \ldots\ldots$ Then by similar triangles

$$y : y' = 1/\beta : 1/\alpha';$$

that is,

Similarly

$$\left.\begin{aligned} \beta y &= \alpha' y'. \\ \beta' y' &= \alpha'' y'', \\ &\ldots\ldots\ldots\ldots \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

First, we shall confine our attention to *three* refracting surfaces. The problem before us is to find the variation of β'' due to the apertures of the successive surfaces.

Now β'' may be regarded as a function of α'' and of the aperture y''. The variation of β'' due to the aperture y'' is of the form $\kappa'' y''^2$. Hence

$$d\beta'' = \kappa'' y''^2 + \left(\frac{d\beta''}{d\alpha''}\right) d\alpha'',$$

where the differential coefficient is partial.

The aperture y'' may be expressed in terms of the first aperture y by the equations (3), and we find

$$\kappa'' y''^2 = \kappa'' \left(\frac{\beta'}{\alpha''}\right)^2 y'^2 = \kappa'' \left(\frac{\beta\beta'}{\alpha'\alpha''}\right)^2 y^2.$$

Also differentiating the equations (1) and (2), we get

$$\frac{d\beta''}{d\alpha''} = \frac{\mu''}{\mu'''}, \qquad \frac{d\alpha''}{\alpha''^2} = \frac{d\beta'}{\beta'^2}.$$

Substituting these values in the expression for $d\beta''$, it becomes

$$d\beta'' = \kappa'' \left(\frac{\beta\beta'}{\alpha'\alpha''}\right)^2 y^2 + \frac{\mu''}{\mu'''} \left(\frac{\alpha''}{\beta'}\right)^2 d\beta'.$$

The variation of β' may be shown in a similar manner to be

$$d\beta' = \kappa' y'^2 + \left(\frac{d\beta'}{d\alpha'}\right) d\alpha'$$

$$= \kappa' \left(\frac{\beta}{\alpha'}\right)^2 y^2 + \frac{\mu'}{\mu''}\left(\frac{\alpha'}{\beta}\right)^2 d\beta;$$

and, lastly, $\qquad d\beta = \kappa y^2.$

Hence, making these substitutions in the value of $d\beta''$, we find

$$\mu''' \, d\beta'' = y^2 \left[\mu' \kappa \left(\frac{\alpha'\alpha''}{\beta\beta'}\right)^2 + \mu'' \kappa' \left(\frac{\beta\alpha''}{\alpha'\beta'}\right)^2 + \mu''' \kappa'' \left(\frac{\beta\beta'}{\alpha'\alpha''}\right)^2\right].$$

The value of κ has been determined by a previous investigation to be

$$\kappa = \frac{1}{2} \frac{1}{\left(\frac{\mu'}{\mu} - 1\right)^2} (\beta - \alpha)^2 \left(\beta - \frac{\mu'\alpha}{\mu}\right),$$

and therefore $\quad \mu'\kappa = \frac{1}{2}\frac{\mu\mu'}{(\mu' - \mu)^2}(\beta - \alpha)^2 \left(\frac{\beta}{\mu'} - \frac{\alpha}{\mu}\right),$

and there are similar expressions for $\mu''\kappa'$ and $\mu'''\kappa''$.

If there are $(n+1)$ refracting surfaces, the corresponding formula will be

$$\mu^{(n+1)} d\beta^{(n)} = y^2 \left[\mu'\kappa \left(\frac{\alpha'\alpha''\alpha''' \dots \alpha^{(n)}}{\beta\beta'\beta'' \dots \beta^{(n-1)}}\right)^2 + \mu''\kappa' \left(\frac{\beta\alpha''\alpha''' \dots \alpha^{(n)}}{\alpha'\beta'\beta'' \dots \beta^{(n-1)}}\right)\right.$$

$$\left. + \mu'''\kappa'' \left(\frac{\beta\beta'\alpha''' \dots \alpha^{(n)}}{\alpha'\alpha''\beta'' \dots \beta^{(n-1)}}\right)^2 + \dots + \mu^{(n+1)}\kappa^{(n)} \left(\frac{\beta\beta'\beta'' \dots \beta^{(n-1)}}{\alpha'\alpha''\alpha''' \dots \alpha^{(n)}}\right)^2\right].$$

130. Exactly the same argument may be applied to a series of thin lenses. If α, β, be the reciprocals of the distances of the point and its first image from the first lens, and ϕ the focal length of the lens, with similar notation for the subsequent lenses, the equations of condition are

$$\beta - \alpha = \phi,$$

$$\beta' - \alpha' = \phi',$$

....

$$\frac{1}{\beta} - \frac{1}{\alpha'} = \frac{1}{\theta},$$

where $1/\theta$ is the distance between the first and second lenses,

with similar equations. Then the final result for n lenses will be

$$d\beta^{(n-1)} = y^2\left[\left(\frac{\alpha'\alpha''\alpha'''\ldots\alpha^{(n-1)}}{\beta\beta'\beta''\ldots\beta^{(n-2)}}\right)^2\kappa + \left(\frac{\beta\alpha''\alpha'''\ldots\alpha^{(n-1)}}{\alpha'\beta'\beta''\ldots\beta^{(n-2)}}\right)^2\kappa' + \ldots\right.$$
$$\left. + \left(\frac{\beta\beta'\beta''\ldots\beta^{(n-2)}}{\alpha'\alpha''\alpha'''\ldots\alpha^{(n-1)}}\right)^2\kappa^{(n-1)}\right].$$

The values of κ, $\kappa'\ldots$ have been already determined in the previous investigations; they are

$$\kappa = \tfrac{1}{2}m(\beta-\alpha)\left[n(\beta-\alpha)^2 - \alpha\beta + \epsilon^2\right],$$

and similar values; and the curvatures of the lenses are given by the equations

$$\left.\begin{aligned}\rho &= p\beta + q\alpha + m\epsilon \\ \rho' &= p\alpha + q\beta + m\epsilon\end{aligned}\right\}$$

and others of similar form.

When the lenses are in contact, $\alpha' = \beta$, $\alpha'' = \beta'$, &c., so that the coefficients of κ, $\kappa'\ldots$ are all unity, and

$$d\beta^{(n-1)} = y^2\{\kappa + \kappa' + \ldots + \kappa^{(n-1)}\}.$$

Thus, to make a system of lenses in contact aplanatic, we must make

$$\kappa + \kappa' + \ldots + \kappa^{(n-1)} = 0;$$

which imposes *one* condition upon the unknown quantities ϵ, $\epsilon'\ldots$ $\epsilon^{(n-1)}$. The lenses may be therefore chosen so as to satisfy $(n-1)$ other conditions.

131. We shall consider the case of *two* lenses in contact, more in detail. The condition of aplanatism is

$$\kappa + \kappa' = 0.$$

Substituting the values of κ, κ', and noting that $\beta - \alpha = \phi$, $\beta' - \alpha' = \phi'$, this equation becomes

$$m\phi\{\epsilon^2 - \alpha\beta + n\phi^2\} + m'\phi'\{\epsilon'^2 - \alpha'\beta' + n'\phi'^2\} = 0.$$

There are therefore two arbitrary quantities ϵ, ϵ' with only one equation to determine them. We may therefore introduce one other condition. The most usual condition in practice is to make the adjacent surfaces of the lenses of equal curvature, the one being convex and the other concave, so that the two lenses may be

cemented together. The condition for this is that $\rho' = \rho''$, using the same notation as before. This condition when expressed in terms of ϵ, ϵ' is equivalent to

$$p\alpha + q\beta + m\epsilon = p'\beta' + q'\alpha' + m'\epsilon';$$

and this equation combined with the previous equation of condition serves completely to determine ϵ, ϵ' and therefore the curvatures of all the surfaces of the lenses.

132. There are however objections to be raised against this plan; the compound lens will be liable to distortion on a change of temperature, if the two glasses expand unequally under the influence of heat. It has therefore been proposed that the other condition should be $d(\kappa + \kappa') = 0$, so that the lens may be aplanatic not only for a particular value of α, but also when this value of α undergoes a small variation.

To develop the consequences of this equation, we must substitute the values of κ, κ' in the equation

$$\frac{d\kappa}{d\alpha} + \frac{d\kappa'}{d\alpha} = 0.$$

Then, noticing that $d\alpha = d\beta = d\alpha' = d\beta'$, it becomes

$$m\phi\left\{2\epsilon\frac{d\epsilon}{d\alpha} - (\alpha + \beta)\right\} + m'\phi'\left\{2\epsilon'\frac{d\epsilon'}{d\alpha'} - (\alpha' + \beta')\right\} = 0.$$

Now if we differentiate the values of ρ, ρ'', in terms of α, β, α', β', ϵ, ϵ', we get

$$\left.\begin{aligned} m\frac{d\epsilon}{d\alpha} + p + q = 0 \\ m'\frac{d\epsilon'}{d\alpha} + p' + q' = 0 \end{aligned}\right\}.$$

These values of $m\dfrac{d\epsilon}{d\alpha}$, $m'\dfrac{d\epsilon'}{d\alpha}$ must be substituted in the preceding equation. For brevity, let

$$2(p + q) = 4\frac{\mu + 1}{\mu + 2} = l,$$

$$2(p' + q') = 4\frac{\mu' + 1}{\mu + 2} = l';$$

and then the equation becomes

$$\phi \{l\epsilon + m(\alpha + \beta)\} + \phi' \{l'\epsilon' + m'(\alpha' + \beta')\} = 0.$$

The quantities ϵ, ϵ' are therefore completely determined, and from them the values of the curvatures of the different surfaces of the lenses may be found.

133. When the incident rays are parallel,

$$\alpha = 0, \quad \beta = \alpha' = \phi, \quad \beta' = \phi + \phi'$$

and therefore the equations of aplanatism are

$$m\phi \{\epsilon^2 + n\phi^2\} + m'\phi \{\epsilon'^2 - \phi(\phi + \phi') + n\phi'^2\} = 0,$$
$$\phi \{l\epsilon + m\phi\} + \phi' \{l'\epsilon' + m'(2\phi + \phi')\} = 0.$$

From these equations ϵ, ϵ' may be found, and the resulting values substituted in the equations

$$\rho = p\phi + m\epsilon, \quad \rho'' = q'\phi + p'(\phi + \phi') + m'\epsilon',$$
$$\rho' = q\phi + m\epsilon, \quad \rho''' = p'\phi + q'(\phi + \phi') + m'\epsilon',$$

and then the compound lens is completely determined, and will be aplanatic, not only for parallel incident rays, but also for rays diverging from a point whose distance is finite and considerable.

We notice that these equations determining the radii of the surfaces do not involve any relation between the *focal lengths* of the two component lenses, and may be satisfied whatever the values of ϕ and ϕ'.

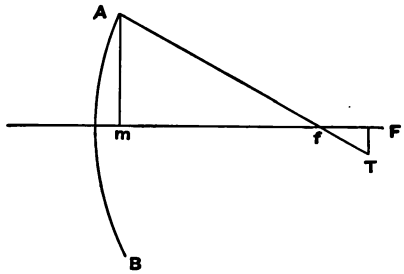
134. So far, the quantity whose value we have been seeking is the variation of the reciprocal of the distance of the intersection of the emergent ray with the axis, arising from the aperture. If we denote this quantity by $-Ky^2$, y being the semi-aperture of the first lens, and if $\alpha' = 1/x'$, we get

$$d\left(\frac{1}{x'}\right) = -Ky^2,$$

and therefore $$dx' = K \,.\, x'^2 y^2.$$

This quantity dx' is the longitudinal aberration, and its value is known when we substitute for K its value as determined by the preceding articles.

Let Af be the extreme ray, meeting the axis in f, and the perpendicular to the axis erected at the geometrical focus F, in T,



then Ff is the longitudinal aberration dx', and FT is the lateral aberration. Draw Am perpendicular to the axis, so that Am may be the aperture of the last lens. Then, by similar triangles,

$$FT : Ff = Am : mf$$
$$= y' : x', \text{ nearly,}$$

where y' is the semi-aperture of the last lens.

And if α, β denote the reciprocals of the distances of the point and its first image from the first lens, α', β' the corresponding quantities for the second lens and so on,

$$\frac{y'}{y} = \frac{\beta\beta'\beta''\ldots}{\alpha\alpha'\alpha''\ldots} = m, \text{ say.}$$

Accordingly, if we substitute the values of Ff and y', we get the value of the lateral aberration,

$$FT = mKx'y^3.$$

The magnitude and position of the least circle of aberration have already been found; it was shown that the radius of this circle is one-fourth of the lateral aberration of the extreme ray, and that the distance of its centre from the geometrical focus is three-fourths of the longitudinal aberration of the extreme ray.

EXAMPLES.

1. Prove that the aberration of a pencil of parallel rays incident directly upon a spherical refracting surface is less than it would be for a reflecting surface of the same shape if the index of refraction be greater than 2.

2. The aberration of a ray which passes through a plate of thickness t is

$$t\left\{1-\frac{\cos\phi}{\sqrt{\mu^2-\sin^2\phi}}\right\},$$

ϕ being the angle of incidence; hence, show that if $\mu^2=2$ there is no aberration, to the third order of small quantities.

3. A pencil is refracted through a sphere of refractive index μ. Show that if y be the breadth of the incident pencil, and f the focal length of the sphere, the formula connecting the conjugate foci, the centre of the sphere being the origin, is

$$\frac{1}{p}-\frac{1}{q}=\frac{1}{f}+\left\{\frac{\mu}{(\mu-1)^2}-\left(\frac{2f}{p}-1\right)^2\right\}\frac{y^2}{8f^2}.$$

4. A pencil of rays is refracted directly through a hemisphere, the distance of the origin from the plane surface, which is that of first incidence, being r/μ^2; show that the aberration of a ray incident at a distance y from the axis is $(\mu^2-1)\,\mu^4y^2/r$, r being the radius.

5. A mirror of given aperture and focal length and of small curvature has the form of a prolate spheroid; show that the aberration for parallel rays varies inversely as the major axis.

6. A ray from Q, on the axis of a parabola, is incident on the curve at P and is reflected so that the reflected ray cuts the axis in q, show that the aberration is equal to

$$\frac{2}{SQ}\left(AS\tan\frac{ASP}{2}\right)^2,$$

where A is the vertex, and S the focus of the parabola.

7. If the law of refraction were assumed to be $\phi=\mu\phi'$, show that the approximate error in finding the point where a ray incident near and parallel to the axis of a spherical refracting surface cuts the axis after refraction would be $\dfrac{1}{6\mu}\dfrac{\mu+1}{\mu-1}\dfrac{y^2}{r}$, with the usual notation.

CHAPTER VIII.

ON THE GENERAL FORM AND PROPERTIES OF A THIN PENCIL. GENERAL REFRACTION OF THIN PENCILS.

135. THE thin pencil here considered is one which has one and only one ray passing through any given point, and is such that no ray is far from a fixed ray, which we call the *principal ray*, and the inclinations of all the rays to the principal ray are small quantities of the first order. The theory is due to Kummer.

We shall take the principal ray as the axis of z. Let (α, β, γ) be the direction cosines of any consecutive ray of the pencil, and $(x, y, 0)$ the co-ordinates of the point where it meets the plane of (x, y); then, by supposition, α and β are small quantities of the first order, and $\gamma = 1$, nearly. Now since one ray, and one only, passes through any point, the co-ordinates x, y are functions of the direction cosines α, β, and we may expand them in terms of α, β by Maclaurin's Theorem. Neglecting powers of α, β higher than the first, the equations become

$$\left.\begin{aligned} x &= \frac{dx}{d\alpha}\alpha + \frac{dx}{d\beta}\beta \\ y &= \frac{dy}{d\alpha}\alpha + \frac{dy}{d\beta}\beta \end{aligned}\right\},$$

or

$$\left.\begin{aligned} x &= e\alpha + f\beta \\ y &= f'\alpha + g\beta \end{aligned}\right\}, \text{ say,}$$

where e, f, f', g are constants depending on the nature of the pencil.

136. In general, consecutive rays do not intersect, but we can find the point on the principal ray which is at the shortest

distance from any consecutive ray; and it will be shown that *these points of shortest distance all lie within fixed limits on the principal ray.* For let z be the distance from the origin of the point on the principal ray which is at the shortest distance from the ray (α, β, γ); and let (λ, μ, ν) be the direction cosines of the shortest distance between the two rays. Then the direction of the line (λ, μ, ν) is perpendicular to both the rays, and therefore

$$\left.\begin{aligned} \nu &= 0 \\ \lambda\alpha + \mu\beta &= 0 \end{aligned}\right\}.$$

The equation to a plane through the ray (α, β, γ) and the shortest distance is

$$\begin{vmatrix} \xi - x, & \eta - y, & \zeta \\ \alpha \;, & \beta \;, & \gamma \\ \lambda \;, & \mu \;, & 0 \end{vmatrix} = 0,$$

or $$-(\xi - x)\,\mu\gamma + (\eta - y)\,\lambda\gamma + \zeta\,(\alpha\mu - \lambda\beta) = 0.$$

This plane will meet the principal ray in the required point of shortest distance. Putting, therefore, $\xi = 0$, $\eta = 0$, in this equation, we get

$$z\,(\alpha\mu - \lambda\beta) = \lambda y - \mu x.$$

We can eliminate λ and μ by means of the equation $\lambda\alpha + \mu\beta = 0$, and we then get

$$\begin{aligned} z\,(\alpha^2 + \beta^2) &= -\,(\alpha x + \beta y) \\ &= -\,\{e\alpha^2 + (f + f')\,\alpha\beta + g\beta^2\}. \end{aligned}$$

If δ denote the angle made with the plane of xz, by a plane through the principal ray and parallel to the ray (α, β, γ), we must have $\tan\delta = \beta/\alpha$, and therefore

$$z = -\,\{e\cos^2\delta + (f + f')\sin\delta\cos\delta + g\sin^2\delta\}.$$

This can be made to assume a simpler form by getting rid of the middle term. Thus, let $\delta = \omega + \phi$, where ω is a fixed angle still to be determined. Then

$$\begin{aligned} z = &-\cos^2\phi\,\{e\cos^2\omega + (f + f')\sin\omega\cos\omega + g\sin^2\omega\} \\ &-\sin^2\phi\,\{e\sin^2\omega - (f + f')\sin\omega\cos\omega + g\cos^2\omega\} \\ &-\sin\phi\cos\phi\,\{(f + f')\cos 2\omega - (e - g)\sin 2\omega\}. \end{aligned}$$

Choose ω so that

$$\left.\begin{aligned}(e-g) &= k\cos 2\omega\\ (f+f') &= k\sin 2\omega\end{aligned}\right\},$$

and therefore $$k^2=(e-g)^2+(f+f')^2.$$

The coefficient of $\sin\phi\cos\phi$ then disappears; the coefficient of $\cos^2\phi$ becomes $-\frac{1}{2}\{e(1+\cos 2\omega)+(f+f')\sin 2\omega+g(1-\cos 2\omega)\}$, or $-\frac{1}{2}(e+g+k)$, and the coefficient of $\sin^2\phi$ may be reduced in the same way. Thus the equation takes the form

$$z=r_1\cos^2\phi+r_2\sin^2\phi$$

where

$$r_1=-\tfrac{1}{2}(e+g+k),$$

$$r_2=-\tfrac{1}{2}(e+g-k).$$

Thus the point of shortest distance lies between the fixed limits, determined by the equations $z=r_1$, $z=r_2$. These points are therefore called the *limiting points* of the ray, and the corresponding planes parallel to two consecutive rays, namely, the planes $\phi=0$, $\phi=\frac{1}{2}\pi$, are called the *principal planes* of the ray.

137. *There are two rays of the system whose shortest distances from the principal ray vanish*, to a first approximation. The length of the shortest distance between the ray (α, β, γ) and the principal ray, is $\lambda x+\mu y$. This will therefore vanish, to a first approximation, if

$$\lambda(e\alpha+f\beta)+\mu(f'\alpha+g\beta)=0.$$

Eliminating the ratio $\lambda:\mu$, and expressing α, β in terms of δ as before, this becomes

$$f'\cos^2\delta+(g-e)\sin\delta\cos\delta-f\sin^2\delta=0,$$

or $$(g-e)\sin 2\delta+(f+f')\cos 2\delta=f-f'.$$

If we substitute $\omega+\phi$ for δ, this equation takes the form

$$\cos 2\phi\{(f+f')\cos 2\omega+(g-e)\sin 2\omega\}$$
$$+\sin 2\phi\{(g-e)\cos 2\omega-(f+f')\sin 2\omega\}=f-f';$$

that is,

$$-\sin 2\phi\,.\,k=f-f'.$$

Hence if ϵ be defined by the equation $\sin\epsilon=(f'-f)/k$,

$$\phi=\frac{\epsilon}{2}\text{ or }\frac{\pi}{2}-\frac{\epsilon}{2}.$$

Hence *there are two planes through the principal ray, which also contain a consecutive ray.* These planes are called the *focal planes* of the ray, and the points where the consecutive rays meet the principal ray are called the two *focal points* of the ray. *The focal planes lie symmetrically with respect to the principal planes;* or, in other words, the plane bisecting the angle between the principal planes also bisects the angle between the focal planes.

Let ρ_1, ρ_2 be the focal distances from the origin. Then

$$\left.\begin{array}{l}\rho_1 = r_1 \cos^2\phi + r_2 \sin^2\phi \\ \rho_2 = r_1 \sin^2\phi + r_2 \cos^2\phi\end{array}\right\},$$

where
$$\sin 2\phi = \frac{f'-f}{k};$$

so that
$$\rho_1 + \rho_2 = r_1 + r_2.$$

This proves that *the focal points lie symmetrically placed with respect to the limiting points;* or, in other words, the point midway between the limiting points is also midway between the focal points. By the properties of the limiting points it follows that the focal points lie between the limiting points.

138. The focal distances may be found more easily by a different method. If the ray through $(x, y, 0)$ meet the plane $z = \rho$ in the point (ξ, η) we must have

$$\left.\begin{array}{l}\xi = x + \rho\alpha = (e+\rho)\,\alpha + f\beta \\ \eta = y + \rho\beta = f'\alpha + (g+\rho)\beta\end{array}\right\}.$$

If now the point $z = \rho$ be a focal point, the consecutive ray meets the principal ray; so that $\xi = 0$, $\eta = 0$. This gives the equations

$$\left.\begin{array}{l}(e+\rho)\,\alpha + f\beta = 0 \\ f'\alpha + (g+\rho)\,\beta = 0\end{array}\right\};$$

and eliminating the ratio $\alpha : \beta$ between them, we get a quadratic equation to determine the focal distances,

$$(e+\rho)(g+\rho) = ff'.$$

139. We shall next prove that *all the rays pass through two fixed lines, which cut the principal ray at right angles in the focal points.* These lines are called *focal lines*, and the first focal line

lies in the second focal plane, and the second focal line in the first focal plane.

To prove this, we find the point of intersection of any ray with the plane $z = \rho_1$. The co-ordinates of such a point of intersection are

$$\left.\begin{aligned} \xi &= x + \rho_1 \alpha \\ \eta &= y + \rho_1 \beta \\ \zeta &= \rho_1 \end{aligned}\right\}.$$

If we draw a plane through the principal ray and this point, we shall find that the position of the plane is independent of the particular ray. For the equation of the plane is

$$\xi (y + \rho_1 \beta) = \eta (x + \rho_1 \alpha),$$

or

$$\xi \{(e + \rho_1) \alpha + f\beta\} = \eta \{f' \alpha + (g + \rho_1)\beta\}.$$

And, by virtue of the equation

$$(e + \rho)(g + \rho) = ff',$$

$$\frac{e + \rho_1}{f'} = \frac{f}{g + \rho_1} = m, \text{ say};$$

then the equation of the plane becomes

$$\xi m = \eta,$$

which is independent of α, β.

This shows that all the rays meet the plane $z = \rho_1$ in points lying along a line through the focal point; this we shall call the first focal line. Similarly, it may be shown that all the rays pass through a similar second focal line passing through the other focus.

It is now clear that the first focal line lies in the second focal plane, because this plane contains two rays meeting at the second focus, and each of these rays intersects the focal line. Similarly, the second focal line lies in the first focal plane.

140. There is a theory of the *density* of the rays, which is analogous to Gauss' measure of the curvature of surfaces. The density of the rays in any section of the pencil perpendicular to the principal ray, may be defined in the following manner. Suppose the plane section of the pencil to be bounded by a small curvi-

linear area. If lines be drawn from the centre of a sphere of unit radius, parallel to the rays of the pencil all round the boundary, they will meet the surface of the sphere in a small closed curve. The density of the rays is defined to be the ratio of the area of the spherical element, to the area of the plane section of the pencil.

Let the section of the pencil be taken by a plane $z = R$; then if the ray through the point $(x, y, 0)$ meet this plane in (ξ, η),

$$\left.\begin{aligned} \xi &= x + R\alpha = (e+R)\alpha + f\beta \\ \eta &= y + R\beta = f'\alpha + (g+R)\beta \end{aligned}\right\}$$

and therefore

$$\eta d\xi - \xi d\eta = (\beta d\alpha - \alpha d\beta)\{(e+R)(g+R) - ff'\}.$$

Integrating this equation, we find

$$\int(\eta d\xi - \xi d\eta) = \int(\beta d\alpha - \alpha d\beta)\{(e+R)(g+R) - ff'\}.$$

Now $\int(\beta d\alpha - \alpha d\beta)$ is the area of the small curve on the sphere of unit radius, and $\int(\eta d\xi - \xi d\eta)$ is the area of the section of the pencil; hence, if the density be denoted by δ, the last equation may be written

$$\frac{1}{\delta} = R^2 + R(e+g) + eg - ff',$$

or

$$\frac{1}{\delta} = (R - \rho_1)(R - \rho_2)$$

where ρ_1, ρ_2 are the distances of the focal points.

141. For optical purposes the most important case is that in which the rays of the pencils are a set of normals to a surface. We must now enquire what modifications of the preceding results are necessary in this case.

The condition that the rays may be cut at right angles by a surface, is that $\alpha dx + \beta dy$ may be a perfect differential. This condition will be satisfied, if

$$\frac{d\alpha}{dy} = \frac{d\beta}{dx}.$$

If we differentiate x and y, we get the equations

$$\left.\begin{aligned} 1 &= \frac{dx}{d\alpha}\frac{d\alpha}{dx} + \frac{dx}{d\beta}\frac{d\beta}{dx} \\ 0 &= \frac{dy}{d\alpha}\frac{d\alpha}{dx} + \frac{dy}{d\beta}\frac{d\beta}{dx} \end{aligned}\right\}$$

$$\left.\begin{aligned} 0 &= \frac{dx}{d\alpha}\frac{d\alpha}{dy} + \frac{dx}{d\beta}\frac{d\beta}{dy} \\ 1 &= \frac{dy}{d\alpha}\frac{d\alpha}{dy} + \frac{dy}{d\beta}\frac{d\beta}{dy} \end{aligned}\right\}.$$

From the first pair of these equations, we find

$$\frac{d\beta}{dx}J = -\frac{dy}{d\alpha};$$

and from the second pair,

$$\frac{d\alpha}{dy}J = -\frac{dx}{d\beta},$$

where J is the Jacobian $\dfrac{d(x, y)}{d(\alpha, \beta)}$.

The condition that the rays may be normal to a surface, therefore, becomes

$$\frac{dx}{d\beta} = \frac{dy}{d\alpha},$$

or in the notation we have employed, $f = f'$.

The equation for determining the azimuths of the focal planes becomes $\sin 2\phi = 0$, which gives $\phi = 0$ or $\frac{1}{2}\pi$. The focal points therefore coincide with the limiting points, and the focal planes are at right angles to each other. *Every ray, therefore, of a system of normals, passes through two straight lines, each perpendicular to the principal ray, and lying in planes through the principal ray, which are perpendicular to each other.*

The two focal points coincide if $f = 0, f' = 0$; and in this case all the lines of the small pencil meet in a point.

142. In order to form an idea of the character of a small pencil, it will be useful to find the equation of the bounding surface of the pencil in some particular cases.

We shall suppose that the pencil meets the orthogonal surface in a small ellipse, whose axes are in the principal planes. For axis of z, we choose as before the principal ray, and the point where this ray meets the orthogonal surface shall be the origin, and the primary and secondary focal planes shall be the planes of xz and yz, respectively. Then if the focal distances be denoted

by v_1, v_2, the equations of the bounding curves of the pencil are

$$\left.\begin{aligned}\frac{x^2}{a^2}+\frac{y^2}{b^2}&=1\\ z&=0\end{aligned}\right\}\text{..........................(1)},$$

$$\left.\begin{aligned}z&=v_1\\ x&=0\end{aligned}\right\}\text{...............(2)},\qquad \left.\begin{aligned}z&=v_2\\ y&=0\end{aligned}\right\}\text{...............(3)}.$$

Every generator of the bounding surface must meet these bounding curves. Let the equations of any generator of the bounding surface be

$$\left.\begin{aligned}x&=Az+B\\ y&=Cz+D\end{aligned}\right\}.$$

Then, since this line intersects the curves (1), (2) and (3), A, B, C, D must satisfy the relations

$$\left.\begin{aligned}&\frac{B^2}{a^2}+\frac{D^2}{b^2}=1\\ &Av_1+B=0\\ &Cv_2+D=0\end{aligned}\right\}.$$

The last two equations serve to determine A and C in terms of B and D, and the equations of the generator may be written

$$x=B\left(1-\frac{z}{v_1}\right)$$

$$y=D\left(1-\frac{z}{v_2}\right).$$

If now we eliminate the remaining constants B, D, the equation to the bounding surface is obtained in the form

$$\frac{x^2}{a^2\left(1-\dfrac{z}{v_1}\right)^2}+\frac{y^2}{b^2\left(1-\dfrac{z}{v_2}\right)^2}=1.$$

143. By giving z different values, we may determine the form of the bounding curves of normal sections of the pencil. These will in general be ellipses, but they may be circular. They will be circular if z be chosen so as to satisfy the equation

$$a^2\left(1-\frac{z}{v_1}\right)^2=b^2\left(1-\frac{z}{v_2}\right)^2.$$

Thus there will be two circular sections of the pencil, one of which always lies between the focal lines. For suppose that v_2 is greater than v_1, then if we take

$$a\left(\frac{z}{v_1}-1\right)=b\left(1-\frac{z}{v_2}\right)=r,$$

we get a circular section, and its position and radius are given by the equations

$$\left.\begin{aligned}\frac{a+b}{z}&=\frac{a}{v_1}+\frac{b}{v_2}\\ \frac{v_2-v_1}{r}&=\frac{v_1}{a}+\frac{v_2}{b}\end{aligned}\right\}.$$

It is easy to see from the first of these equations, that z lies between v_1 and v_2.

This circle is sometimes called the *circle of least confusion*, as will be explained later.

144. But every small pencil has not a circular section of this form; the preceding argument only applies to symmetrical pencils. For instance, if the curve in which the pencil cuts the orthogonal curve be an ellipse whose principal axes do not lie in the focal plane, we may take as the equation of the bounding curve

$$\left.\begin{aligned}ax^2+2hxy+by^2&=1\\ z&=0\end{aligned}\right\},$$

in which case the equation of the bounding surface of the pencil may be shown to be

$$\frac{ax^2}{\left(1-\frac{z}{v_1}\right)^2}+\frac{2hxy}{\left(1-\frac{z}{v_1}\right)\left(1-\frac{z}{v_2}\right)}+\frac{by^2}{\left(1-\frac{z}{v_2}\right)^2}=1.$$

Whatever constant value we may give to z, the equation to the section can never take the form of a circle.

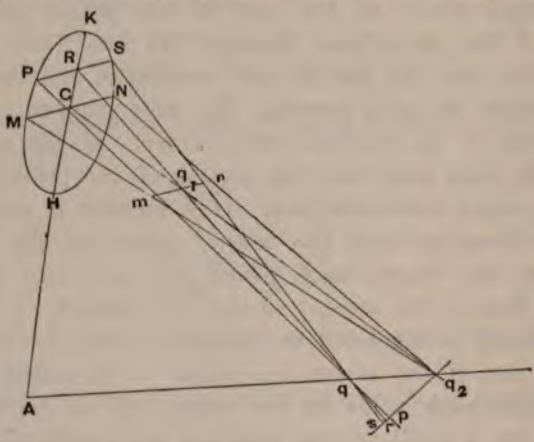
In the case of an optical instrument, however, the orthogonal surfaces are always surfaces of revolution about the axis of the instrument, and the first focal plane of a small part of a symmetrical pencil will be a meridian plane, so that the small pencil has the necessary symmetry, and therefore has a circular section.

145. It has been seen that when a small pencil emerges from an optical system, it will not in general have a single focus through which all the rays of the pencil pass, but all the rays of the pencil pass through two focal lines; we must next enquire into the nature and position of the image of an object afforded by such pencils. We may suppose one of the small pencils to be received on a screen. If the screen pass through the first focal line, the section of the pencil will be a small line, say a horizontal line. If a number of such pencils, originally diverging from the several points of the object be received on the screen, each pencil will give rise to a short horizontal line on the screen; so that the breadth of the image will be greatly exaggerated as compared with its length. In the same way, if the screen be placed at the other focal line of the pencil, each point on the object will be represented in the image by a small vertical line, and the length of the image will be exaggerated as compared with its breadth. Both these images are defective, in that they distort the shape of the object. The best image is given when the screen occupies the position of the circle of least confusion; for then, corresponding to each point of the object, there appears a small circular patch of light on the screen, and if the circle of least confusion be very small, this will not seriously impair the value of the image. The image is therefore taken to be the aggregation of the overlapping circles of least confusion. The size of the circle of least confusion may be taken to represent a measure of the indistinctness.

146. The nature of a small pencil may be investigated in a less satisfactory manner, by Elementary Geometry. When the system of rays is symmetrical about an axis, we have seen that the orthogonal surface is a surface of revolution. We are now going to consider a small pencil of the system whose rays meet the orthogonal surface within a small area. We shall suppose this bounding line of the pencil at the orthogonal surface to be a small ellipse of semi-axes a, b, whose centre is C.

Let MCN, PRS... be lines of curvature, of the system which consists of circles whose centres lie on the axis. Then rays from all points of the line MCN will meet the axis in the same

point q_2; and since the area is small, MCN may be taken to be a straight line; and therefore all rays passing through the



line MCN, lie in the plane q_2MN and meet in q_2. Similarly all the rays which meet the orthogonal surface in the line PRS lie in a plane qPS and meet the axis in a point q. Let these planes meet in the line mq_1n. When the line PRS moves up to coincide with MCN this line mq_1n takes up a definite limiting position; and since the area on the orthogonal surface is small, all the planes such as qPS very nearly pass through this limiting position of the line mq_1n. This line is called the *primary focal line*, and the point q_1, where the principal ray meets it, is called the *primary focus*. Thus all the rays of the pencil pass through the primary focal line very nearly, and will be treated as if actually passing through it.

147. Also all the rays of the pencil meet the axis, and therefore the axis might be taken as a secondary focal line; but this is not convenient, because this line is not perpendicular to the principal ray of the pencil. A section of the pencil by a plane through q_2 perpendicular to the principal ray does not differ much from a straight line, the actual shape of the section is a curve with two loops, like a slender figure of eight. For since all the rays from MCN pass through q_2, the breadth of the section at q_2 vanishes. But the rays from the line PRS meet in q, and therefore they will have diverged again before meeting

the plane of section, giving a section of small breadth prs; and similarly, if we consider a line of curvature below MCN, the rays from it meet on the axis beyond q_2, and therefore will not have met when they reach the plane of section. It appears therefore that the figure bulges out both above and below q_2; but it will be proved that the breadth of the section is a small quantity of the second order, and the breadth will therefore be neglected, and the section will be considered to be a straight line.

The breadth of the section may be found from the figure by the consideration of similar triangles. Let θ be the inclination to the axis of the principal ray; then

$$\frac{pr}{PR} = \frac{qr}{qR} = \frac{q_2 r \cot\theta}{qR},$$

since the angle qrq_2 is very nearly a right angle.

Also $$\frac{q_2 r}{CR} = \frac{q_1 q_2}{Cq_1} = \frac{v_2 - v_1}{v_1},$$

where v_1, v_2 denote the distances Cq_1, Cq_2, respectively; and therefore, substituting the value of $q_2 r$, we find

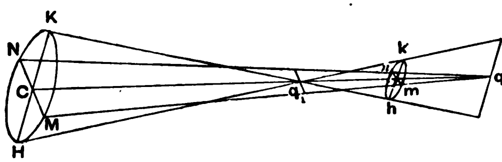
$$pr = \frac{PR \, . \, CR}{qR} \, . \, \frac{v_2 - v_1}{v_1} \cot\theta.$$

The greatest values of PR and CR are respectively b, a, and qR is equal to v_2 nearly, so that pr is of the order $ab\,(v_2 - v_1)/v_1 v_2$. But a, b, are very small compared with v_1, v_2, and therefore the breadth pr is always a small quantity of the second order.

The line is called the *secondary focal line*, and the point q_2 the *secondary focus*. The plane through C and the axis is called the *primary plane*, and the plane through the principal ray, perpendicular to the primary plane, is called the *secondary plane*. Thus the primary plane contains the secondary focal line, and the primary focal line lies in the secondary plane.

148. If we assume that when we choose a section between the two focal lines such that the breadth of the section in the primary plane is equal to that in the secondary plane, this section is circular, the position and magnitude of the circle of least confusion may be found by elementary geometry.

Let *mhnk* be a section perpendicular to the principal ray and let *hok*, *mon* be the breadths in the primary and secondary planes,



respectively, and suppose each of these is equal to $2r$, and let $z = Co$. Then by similar triangles,

$$\left.\begin{aligned} hk : HK &= oq_1 : Cq_1 \\ \text{and also} \qquad mn : MN &= oq_2 : Cq_2 \end{aligned}\right\};$$

that is,

$$\left.\begin{aligned} \frac{r}{a} &= \frac{z - v_1}{v_1} \\ \frac{r}{b} &= \frac{v_2 - z}{v_2} \end{aligned}\right\}.$$

These equations determine the position and radius of the circle of least confusion, and give the same results as before, namely,

$$\frac{a+b}{z} = \frac{a}{v_1} + \frac{b}{v_2},$$

$$\frac{v_2 - v_1}{r} = \frac{v_1}{a} + \frac{v_2}{b}.$$

When the section of the pencil by the orthogonal surface is circular, then $a = b$, and the formula to find the circular section is

$$\frac{2}{z} = \frac{1}{v_1} + \frac{1}{v_2},$$

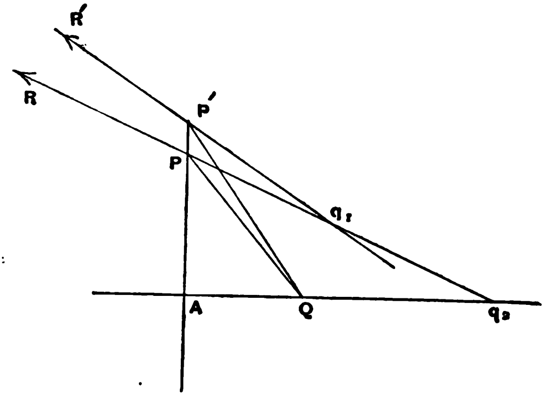
and therefore z is a harmonic mean between v_1 and v_2.

149. We shall now investigate by elementary geometry the positions of the primary and secondary foci of a small pencil diverging from a point, after reflexion or refraction at a plane or spherical surface.

When a small pencil is reflected obliquely at a plane surface, we have seen that the reflected rays all pass accurately through a point, so that the two foci coincide. In this case there is no circle of least confusion; the image of a point is a point, and the definition is perfect.

Next, let the pencil be refracted at a plane surface.

Let QP be the principal ray of the pencil, q_2P the direction of the ray after refraction. Let the angles of incidence and refraction of this ray be, respectively, ϕ and ϕ'. Let QP' be a consecutive ray



in the primary plane and let this ray after refraction meet q_2P in q_1. Then, ultimately, when P' coincides with P, q_1 will be the primary focus. If QA be drawn normal to the plane, and if the refracted ray produced backwards meet QA in q_2, then q_2 will be the secondary focus. Let $QP = u$, $q_1P = v_1$, $q_2P = v_2$.

Then $$\mu \sin \phi = \mu' \sin \phi',$$

and therefore, $$\mu \cos \phi \, d\phi = \mu' \cos \phi' \, d\phi'.$$

Let the line PP' be denoted by x. Then, since $d\phi$ represents the angle PQP',

$$d\phi = \frac{x \cos \phi}{u};$$

and similarly $$d\phi' = \frac{x \cos \phi'}{v_1}.$$

If we substitute these values of $d\phi$, $d\phi'$ in the previous equation and divide both sides by x, it becomes

$$\frac{\mu \cos^2 \phi}{u} = \frac{\mu' \cos^2 \phi'}{v_1} \quad \text{.....................(1).}$$

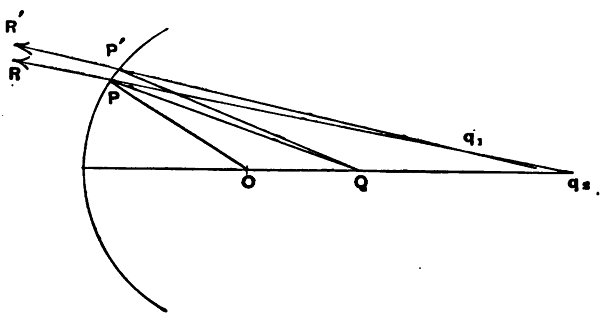
Also $$\sin \phi = \frac{AP}{u},$$

$$\sin \phi' = \frac{AP}{v_2},$$

and therefore $$\frac{\mu}{u} = \frac{\mu'}{v_2} \qquad\qquad (2).$$

The values of v_1 and v_2 are therefore determined.

150. Next, let the pencil be refracted at a spherical surface of radius r.



Let QP be the principal ray, and q_2q_1P its direction after refraction. Let QP' be a consecutive ray in the primary plane; and let the corresponding refracted ray meet q_2P in q_1. Then q_1 will ultimately be the primary focus, and if Pq_1 meet the axis in q_2, q_2 is the secondary focus. Let O be the centre of the sphere, and as before, let $QP = u$, $q_1P = v_1$, $q_2P = v_2$. Let the incident ray QP, the refracted ray q_2P, and the radius OP, make angles χ, χ', θ with the axis, respectively. Then, if ϕ, ϕ' be the angles of incidence and refraction, $\phi = \theta - \chi$, $\phi' = \theta - \chi'$.

By the law of refraction, $\mu \sin\phi = \mu' \sin\phi'$;

and therefore $$\mu \cos\phi \, d\phi = \mu' \cos\phi' \, d\phi'.$$

Expressing $d\phi$, $d\phi'$ in terms of the differentials of θ, χ, χ', this equation becomes

$$\mu \cos\phi \, d\chi - \mu' \cos\phi' \, d\chi' = (\mu \cos\phi - \mu' \cos\phi') \, d\theta.$$

But if we denote the arc PP' by x, it is easy to see that $d\chi = x \cos\phi/u$, $d\chi' = x \cos\phi'/v_1$, and $d\theta = x/r$; and therefore

$$\frac{\mu \cos^2\phi}{u} - \frac{\mu' \cos^2\phi'}{v_1} = \frac{\mu \cos\phi - \mu' \cos\phi'}{r} \qquad\qquad (1).$$

This equation determines the value of v_1.

Also, by equating areas of triangles, we get the identity

$$\Delta\, q_2PO - \Delta\, QPO = \Delta\, q_2PQ;$$

and if we express these areas in terms of the lengths u, v_2, r, this identity becomes

$$v_2 r \sin\phi' - ur\sin\phi = uv_2 \sin(\phi' - \phi).$$

By virtue of the law of refraction, $\mu\sin\phi = \mu'\sin\phi'$, this equation may be expressed in the form

$$\mu v_2 r - \mu' ur = uv_2\{\mu\cos\phi - \mu'\cos\phi'\},$$

or

$$\frac{\mu}{u} - \frac{\mu'}{v_2} = \frac{\mu\cos\phi - \mu'\cos\phi'}{r} \quad\text{..................(2)},$$

and thus the value of v_2 is determined.

151. In the case of reflexion at a spherical surface, we may proceed in exactly the same manner as in the case of refraction; or we may deduce the corresponding formulæ by putting $\mu' = -\mu$. We thus get

$$\left.\begin{aligned}\frac{1}{u} + \frac{1}{v_1} &= \frac{2}{r\cos\phi}\\ \frac{1}{u} + \frac{1}{v_2} &= \frac{2\cos\phi}{r}\end{aligned}\right\}.$$

General Theory of the Refraction of thin pencils.

152. It has been shown that all the rays of a thin pencil, which can be cut at right angles by a surface, pass through two lines, the planes through these lines and the principal ray being perpendicular to each other. Let the axis of the pencil be taken as the axis of z, and let the focal planes be the planes of xz and yz, so that all the rays of the pencil pass through a line $x=0$, $z=a$, and also through a line $y=0$, $z=b$. Let any ray of the pencil meet the plane of xy in a point (x, y), and let the equation of the ray be

$$\frac{\xi - x}{\alpha} = \frac{\eta - y}{\beta} = \frac{\zeta}{\gamma} = r.$$

Then, since it meets the first focal line,

$$\left.\begin{aligned}\alpha r + x &= 0\\ \gamma r &= a\end{aligned}\right\},$$

and therefore

$$\frac{\alpha}{\gamma} = -\frac{x}{a}.$$

Similarly
$$\frac{\beta}{\gamma} = -\frac{y}{b},$$
and therefore
$$\alpha dx + \beta dy + \gamma dz = \gamma\left(dz - \frac{xdx}{a} - \frac{ydy}{b}\right).$$

But $\gamma = 1$, to the approximation to which we have been limiting ourselves, so that if V denote the characteristic function, we have
$$\begin{aligned} dV &= \mu(\alpha dx + \beta dy + \gamma dz) \\ &= \mu\left(dz - \frac{xdx}{a} - \frac{ydy}{b}\right), \end{aligned}$$
and therefore
$$V = K + \mu\left(z - \frac{x^2}{2a} - \frac{y^2}{2b}\right) \quad\text{.................(1).}$$

The characteristic function for a small pencil in any position may be found from this expression by transformation of axes. First turn the axes of x and y about the axis of z through an angle θ from x to y, then the value of the characteristic function becomes
$$V = K + \mu\left(z - \frac{x^2}{2A} - \frac{y^2}{2B} - \frac{xy}{C}\right) \quad\text{...........(2),}$$
where
$$\frac{1}{A} = \frac{\cos^2\theta}{a} + \frac{\sin^2\theta}{b},$$
$$\frac{1}{B} = \frac{\sin^2\theta}{a} + \frac{\cos^2\theta}{b},$$
$$\frac{1}{C} = \left(\frac{1}{a} - \frac{1}{b}\right)\sin\theta\cos\theta.$$

Conversely, if the characteristic function be given in the form (2), the values of θ, a, b may be found by means of these equations; in other words, the directions of the focal planes and the two focal distances may be determined in terms of A, B, C.

Next, turn the axes round the axis of y through an angle ϕ from x towards z; we then deduce
$$V = K + \mu\left[z\cos\phi - x\sin\phi - \frac{(x\cos\phi + z\sin\phi)^2}{2A} - \frac{y^2}{2B} - \frac{y(x\cos\phi + z\sin\phi)}{C}\right] \quad\text{......(3),}$$
and this is the general form of the characteristic function for a small pencil.

153. *To find the relations between the constants A, B, C, θ, ϕ, A', B', C', θ', ϕ' for the incident and refracted pencils.*

Let the normal to the refracting surface be taken as the axis of z, and the plane of incidence as the plane of xz, so that if ϕ be the angle of incidence of the principal ray, the characteristic function for the incident pencil will be of the form (3). The characteristic function for the refracted pencil will be of the same form with dashed letters for the constants. Let the equation to the surface in the immediate neighbourhood of the point of incidence be

$$z = \frac{x^2}{2P} + \frac{y^2}{2Q} + \frac{xy}{R} \dots\dots$$

The characteristic function is continuous at this surface, and therefore $V = V'$ along it. In the equation

$$V - V' = 0,$$

substitute the value of z from the equation to the surface; then neglecting powers of x, y above the second, it becomes

$$K - K' + \left(\frac{x^2}{2P} + \frac{y^2}{2Q} + \frac{xy}{R}\right)(\mu \cos\phi - \mu' \cos\phi') - x(\mu \sin\phi - \mu' \sin\phi')$$

$$-\mu\left(\frac{x^2 \cos^2\phi}{2A} + \frac{y^2}{2B} + \frac{xy \cos\phi}{C}\right) + \mu'\left(\frac{x^2 \cos^2\phi'}{2A'} + \frac{y^2}{2B'} + \frac{xy \cos\phi'}{C'}\right) = 0.$$

This equation must be true for all small values of x, y. Equating to zero the several coefficients, we find

$$K = K',$$

$$\mu \sin\phi = \mu' \sin\phi' \dots\dots\dots\dots\dots\dots\dots (4);$$

$$\left.\begin{aligned} \frac{\mu\cos^2\phi}{A} - \frac{\mu'\cos^2\phi'}{A'} &= \frac{\mu\cos\phi - \mu'\cos\phi'}{P} \\ \frac{\mu}{B} - \frac{\mu'}{B'} &= \frac{\mu\cos\phi - \mu'\cos\phi'}{Q} \\ \frac{\mu\cos\phi}{C} - \frac{\mu'\cos\phi'}{C'} &= \frac{\mu\cos\phi - \mu'\cos\phi'}{R} \end{aligned}\right\} \dots\dots(5).$$

The equation (4) expresses the ordinary law of refraction and determines the angle ϕ'; the equations (5) then determine the values of the constants A', B', C', and from these may be calculated all the circumstances of the refracted pencil.

154. In this investigation the plane of incidence and refraction of the principal ray is taken as the plane of xz. If this plane be also a plane of principal curvature of the refracting surface, we shall have $1/R = 0$. In this case, if the focal lines of the incident pencil be, respectively, in and perpendicular to the plane of incidence, the focal lines of the emergent pencil will also be, respectively, in and perpendicular to the same plane. For if $1/C = 0$, and $1/R = 0$, we must also have $1/C' = 0$. If the refracting surface be a plane or a sphere, the plane of incidence is always a plane of principal curvature; and when the refracting surface is a surface of revolution the plane of incidence is a plane of principal curvature when the principal ray intersects the axis. In these cases, therefore, if the focal lines of the incident pencil lie, respectively, in and perpendicular to the plane of incidence, the focal lines of the emergent pencil will also lie, respectively, in and perpendicular to that plane. The same will be true if the incident pencil diverges from, or converges to a point.

155. If the refracting surface be a sphere, $1/R = 0$, and $P = Q = r$, where r is the radius of the sphere. The equations connecting the constants of the incident and refracted pencils in this case become

$$\frac{\mu \cos^2 \phi}{A} - \frac{\mu' \cos^2 \phi'}{A'} = \frac{\mu \cos \phi - \mu' \cos \phi'}{r},$$

$$\frac{\mu}{B} - \frac{\mu'}{B'} = \frac{\mu \cos \phi - \mu' \cos \phi'}{r},$$

$$\frac{\mu \cos \phi}{C} - \frac{\mu' \cos \phi'}{C'} = 0.$$

If the focal lines of the incident pencil lie, respectively, in and perpendicular to the meridian plane, $1/C = 0$, and therefore $1/C' = 0$, so that the focal lines of the emergent pencil also lie, respectively, in and perpendicular to the meridian plane. The same is true if the incident rays diverge from a point; for in this case $a = b$, so that $1/C = 0$, and $A = B$. If, as before, we denote the distance of the radiant point from the surface by u, and the distance of the foci by v, v', the equations

take the form

$$\frac{\mu\cos^2\phi}{u}-\frac{\mu'\cos^2\phi'}{v}=\frac{\mu\cos\phi-\mu'\cos\phi'}{r},$$

$$\frac{\mu}{u}-\frac{\mu'}{v'}=\frac{\mu\cos\phi-\mu'\cos\phi'}{r},$$

which agrees with the result previously obtained.

156. *To find the focal lines of a small pencil after refraction through a prism.*

We shall suppose the axis of the pencil to lie in a plane perpendicular to both faces of the prism, and that the pencil passes through the prism so close to the edge that the length of its path inside the prism may be neglected. Let ϕ, ϕ' be the angles of incidence and refraction of the principal ray at the first surface, ψ' and ψ the angles of incidence and emergence at the second surface. The plane of incidence is the same for both refractions. Let the characteristic function for the incident pencil, when the axis of z is in the direction of the principal ray, be

$$V=K+\left(z-\frac{x^2}{2A}-\frac{y^2}{2B}-\frac{xy}{C}\right),$$

and let V_1 be the corresponding function for the pencil after one refraction, so that

$$V_1=K+\mu\left(z-\frac{x^2}{2A_1}-\frac{y^2}{2B_1}-\frac{xy}{C_1}\right);$$

and after the second refraction let the function be

$$V'=K+\left(z-\frac{x^2}{2A'}-\frac{y^2}{2B'}-\frac{xy}{C'}\right),$$

the positive direction of the axis of z being in each case opposite to that of the transmitted light. Then, at the first refraction the equations connecting A, B, C, A_1, B_1, C_1, are

$$\left.\begin{aligned}\frac{\cos^2\phi}{A}&=\frac{\mu\cos^2\phi'}{A_1}\\ \frac{1}{B}&=\frac{\mu}{B_1}\\ \frac{\cos\phi}{C}&=\frac{\mu\cos\phi'}{C'}\end{aligned}\right\},$$

and at the second refraction the equations connecting A_1, B_1, C_1, and A', B', C', are

$$\frac{\mu \cos^2 \psi'}{A_1} = \frac{\cos^2 \psi}{A'},$$

$$\frac{\mu}{B_1} = \frac{1}{B'},$$

$$\frac{\mu \cos \psi'}{C_1} = \frac{\cos \psi}{C}.$$

Eliminating A_1, B_1, C_1 between these two sets of equations, we arrive at the equations

$$\frac{1}{A}\,\frac{\cos^2 \phi}{\cos^2 \phi'} = \frac{1}{A'}\,\frac{\cos^2 \psi}{\cos^2 \psi'},$$

$$\frac{1}{B} = \frac{1}{B'},$$

$$\frac{1}{C}\,\frac{\cos \phi}{\cos \phi'} = \frac{1}{C'}\,\frac{\cos \psi}{\cos \psi'}.$$

These equations completely determine all the circumstances of the emergent pencil.

If $1/C = 0$ we must have $1/C' = 0$ also; that is, if the focal lines of the incident pencil be, respectively, parallel and perpendicular to the edge of the prism, the focal lines of the emergent will also be, respectively, parallel and perpendicular to the edge of the prism. If the incident pencil diverge from a point, so that $1/C = 0$, and $A = B$, the emergent pencil will not in general diverge from a point, but from a pair of focal lines, respectively parallel and perpendicular to the refracting edge; but if the principal ray pass through the prism with minimum deviation, we shall have $\phi = \psi$, $\phi' = \psi'$, so that $A = A'$, and therefore $A' = B'$. In this case, the emergent pencil will diverge from a point. The distances of the two foci from the edge of the prism are, respectively, A and A', measured both in the direction of the passage of light; thus the foci are equally distant from the refracting angle of the prism.

157. Next we shall consider the refraction of a small pencil through a thin lens both of whose surfaces are cylindrical, but such that the generating lines of the two cylinders are inclined to each other at an angle 2α.

Let the radii of the surfaces be a, b, and in the case considered, let these radii and all distances be measured in a direction opposite to that of the incident pencil. Let the axis of the lens be taken as the axis of z, and let the plane of yz bisect the acute angle between the generating lines of the two cylinders. Let the circumstances of the pencil be originally expressed in the manner of § 152 by the constants A, B, C, and after one refraction by A_1, B_1, C_1, and after the second refraction by A', B', C'. Then the values of P, Q, R for the first surface are

$$\left.\begin{aligned} P &= \frac{\cos^2\alpha}{a} \\ Q &= \frac{\sin^2\alpha}{a} \\ R &= \frac{\sin\alpha\cos\alpha}{a} \end{aligned}\right\},$$

and for the second surface the corresponding values are

$$\left.\begin{aligned} P' &= \frac{\cos^2\alpha}{b} \\ Q' &= \frac{\sin^2\alpha}{b} \\ R' &= -\frac{\sin\alpha\cos\alpha}{b} \end{aligned}\right\}.$$

Moreover the angles of incidence and refraction are very small, so that $\cos\phi = 1$, $\cos\phi' = 1$ approximately, and therefore the equations connecting the constants are,

$$\left.\begin{aligned} \frac{\mu}{A_1} - \frac{1}{A} &= \frac{(\mu-1)\cos^2\alpha}{a} \\ \frac{\mu}{B_1} - \frac{1}{B} &= \frac{(\mu-1)\sin^2\alpha}{a} \\ \frac{\mu}{C_1} - \frac{1}{C} &= \frac{(\mu-1)\sin\alpha\cos\alpha}{a} \end{aligned}\right\},$$

and also

$$\left.\begin{aligned}\frac{1}{A'}-\frac{\mu}{A_1}&=\frac{(1-\mu)\cos^2\alpha}{b}\\ \frac{1}{B'}-\frac{\mu}{B_1}&=\frac{(1-\mu)\sin^2\alpha}{b}\\ \frac{1}{C'}-\frac{\mu}{C_1}&=-\frac{(1-\mu)\sin\alpha\cos\alpha}{b}\end{aligned}\right\}.$$

If we eliminate A_1, B_1, C_1, by adding the corresponding equations of these sets, we get finally

$$\left.\begin{aligned}\frac{1}{A'}-\frac{1}{A}&=(\mu-1)\cos^2\alpha\left(\frac{1}{a}-\frac{1}{b}\right)\\ \frac{1}{B'}-\frac{1}{B}&=(\mu-1)\sin^2\alpha\left(\frac{1}{a}-\frac{1}{b}\right)\\ \frac{1}{C'}-\frac{1}{C}&=(\mu-1)\sin\alpha\cos\alpha\left(\frac{1}{a}+\frac{1}{b}\right)\end{aligned}\right\}.$$

Now if u, v be the distances of the two focal lines from the lens, and if θ be the angle between the plane of yz and the focal line corresponding to u, measured from the axis of y towards that of x, then

$$\left.\begin{aligned}\frac{1}{A'}&=\frac{\cos^2\theta}{u}+\frac{\sin^2\theta}{v}\\ \frac{1}{B'}&=\frac{\sin^2\theta}{u}+\frac{\cos^2\theta}{v}\\ \frac{1}{C'}&=\left(\frac{1}{u}-\frac{1}{v}\right)\sin\theta\cos\theta\end{aligned}\right\};$$

and if the incident pencil be diverging from a point at a distance x from the lens, then

$$\left.\begin{aligned}\frac{1}{A}&=\frac{1}{x}\\ \frac{1}{B}&=\frac{1}{x}\\ \frac{1}{C}&=0\end{aligned}\right\}.$$

Hence, subtracting the first two equations,

$$\left(\frac{1}{u}-\frac{1}{v}\right)\cos 2\theta=(\mu-1)\cos 2\alpha\left(\frac{1}{a}-\frac{1}{b}\right);$$

also $$\left(\frac{1}{u}-\frac{1}{v}\right)\sin 2\theta=(\mu-1)\sin 2\chi\left(\frac{1}{a}+\frac{1}{b}\right),$$

by the third equation.

A lens of this kind does not bring a pencil radiating from a point to a single focus, and therefore it is said to be *astigmatic*, and the measure of the astigmatism may be taken to be the value of $1/u \sim 1/v$.

To find this value we must square and add the last two equations; and so we obtain

$$\frac{1}{u}-\frac{1}{v}=(\mu-1)\sqrt{\frac{1}{a^2}+\frac{1}{b^2}-\frac{2}{ab}\cos 4\chi}.$$

If the second surface be plane, $1/v=0$, and therefore

$$\frac{1}{u}-\frac{1}{v}=\frac{\mu-1}{a}.$$

Such a lens has been called by Professor Stokes an *astigmatic lens*, and the line through the centre of the lens in the direction of the generators of the cylindrical surface the *astigmatic axis* of the lens.

158. *To find the focal lines of a small pencil after oblique centrical refraction through a thin lens.*

The central ray of the pencil passes through the centre of the lens, and will therefore emerge parallel to its original direction. Let ϕ be the angle of incidence and final emergence of the principal ray, ϕ' the angle of refraction within the lens. The plane of incidence is the same for both refractions. Let the characteristic function for the incident pencil, when the axis of z is in the direction of the incident ray and the plane of xz the plane of incidence, be

$$V=K+\left\{z-\frac{x^2}{2A}-\frac{y^2}{2B}-\frac{z^2}{C}\right\},$$

and let the corresponding functions after one and two refractions be, respectively,

$$V_1=K+\mu\left\{z-\frac{x^2}{2A_1}-\frac{y^2}{2B_1}-\frac{z^2}{C_1}\right\},$$

$$V'=K+\quad\left\{z-\frac{x^2}{2A'}-\frac{y^2}{2B'}-\frac{z^2}{C'}\right\},$$

μ being the refractive index of the substance of the lens, and the direction of the axis of z being in each case opposite to the direction of the transmitted light. Then at the first refraction,

$$\frac{\cos^2\phi}{A} - \frac{\mu\cos^2\phi'}{A_1} = \frac{\cos\phi - \mu\cos\phi'}{r},$$

$$\frac{1}{B} - \frac{\mu}{B_1} = \frac{\cos\phi - \mu\cos\phi'}{r},$$

$$\frac{\cos\phi}{C} - \frac{\mu\cos\phi'}{C_1} = 0,$$

and at the second refraction,

$$\frac{\mu\cos^2\phi'}{A_1} - \frac{\cos^2\phi}{A'} = \frac{\mu\cos\phi' - \cos\phi}{r'},$$

$$\frac{\mu}{B_1} - \frac{1}{B'} = \frac{\mu\cos\phi' - \cos\phi}{r'},$$

$$\frac{\mu\cos\phi'}{C_1} - \frac{\cos\phi}{C'} = 0,$$

where r, r' are the radii of the spherical surfaces.

Eliminating A_1, B_1, C_1, the relations between the constants of the incident and emergent pencil are

$$\cos^2\phi\left\{\frac{1}{A'} - \frac{1}{A}\right\} = (\mu\cos\phi' - \cos\phi)\left\{\frac{1}{r} - \frac{1}{r'}\right\},$$

$$\frac{1}{B'} - \frac{1}{B} = (\mu\cos\phi' - \cos\phi)\left(\frac{1}{r} - \frac{1}{r'}\right),$$

$$\frac{1}{C'} = \frac{1}{C}.$$

159. If the incident pencil diverge from a single point at a distance u from the lens, $1/C = 0$, $A = B = u$. Then $1/C' = 0$, so that the focal lines of the emergent pencil lie, respectively, in and perpendicular to the plane of incidence. If v, v' be their distances from the surface

$$\cos^2\phi\left\{\frac{1}{v} - \frac{1}{u}\right\} = \frac{1}{v'} - \frac{1}{u} = (\mu\cos\phi' - \cos\phi)\left(\frac{1}{r} - \frac{1}{r'}\right).$$

In the case in which ϕ and ϕ' are small, we may substitute

$\cos\phi = 1 - \frac{1}{2}\phi^2$, $\cos\phi' = 1 - \frac{1}{2}\phi'^2$, and by the law of refraction, $\phi = \mu\phi'$, and therefore

$$\cos\phi' = 1 - \frac{\phi^2}{2\mu^2}.$$

Substituting these values for $\cos\phi$ and $\cos\phi'$, we get

$$\mu\cos\phi' - \cos\phi = (\mu - 1)\left\{1 + \frac{\phi^2}{2\mu}\right\}.$$

If, therefore, we call the focal length of the lens f, so that

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{r'}\right),$$

we get
$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}\left\{1 + \phi^2\left(1 + \frac{1}{2\mu}\right)\right\},$$

$$\frac{1}{v'} - \frac{1}{u} = \frac{1}{f}\left\{1 + \frac{\phi^2}{2\mu}\right\}.$$

160. One of the defects of an image of an object as seen through a lens, is that it appears curved, when the object is plane. We proceed to find the curvature of an image close to the axis; in other words, we shall compare the curvatures of sections of the object and its image made by any plane through the axis of the lens. This curvature of the image will depend partly upon the obliquity of the pencils proceeding from the points of the object more remote from the axis.

Let u be the distance of the point of the object under consideration from the centre of the lens, and ϕ the obliquity of the principal ray of the pencil. Then, in general, there are two focal lines at distances v, v', from the centre, where

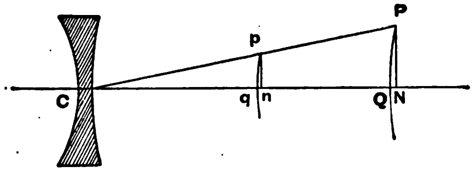
$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}\left\{1 + \phi^2\left(1 + \frac{1}{2\mu}\right)\right\},$$

and
$$\frac{1}{v'} - \frac{1}{u} = \frac{1}{f}\left\{1 + \frac{\phi^2}{2\mu}\right\};$$

so that both foci may be included under the formula

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}(1 + k\phi^2).$$

Let PQ, pq represent sections of the object and its image, respectively, P and p being corresponding points, Q and q being the points where the sections are cut by the axis. Draw the



ordinates PN, pn perpendicular to the axis. Then for the points Q, q the obliquity ϕ is zero, and therefore

$$\frac{1}{Cq} - \frac{1}{CQ} = \frac{1}{f},$$

and for the other corresponding points P, p,

$$\frac{1}{Cp} - \frac{1}{CP} = \frac{1}{f}(1 + k\phi^2).$$

Multiply the former by $\cos\phi$ and subtract the latter, and then we get

$$\frac{qn}{Cp.Cq} - \frac{QN}{CP.CQ} = \frac{1}{f}\left\{1 - \frac{\phi^2}{2} - 1 - k\phi^2\right\}$$

$$= -\frac{1}{f}\left(\frac{1}{2} + k\right)\phi^2.$$

But $Cp \,.\, Cq = Cp^2$, ultimately, and therefore

$$Cp \,.\, Cq\,\phi^2 = \{Cp \,.\, \phi\}^2 = pn^2,$$

and similarly, $$CP \,.\, CQ\,\phi^2 = \{CP \,.\, \phi\}^2 = PN^2,$$

so that we get

$$\frac{qn}{pn^2} - \frac{QN}{PN^2} = -\frac{(2k+1)}{2f}.$$

And if ρ, ρ' be the radii of curvature of the object and image, respectively, by Newton's theory of curvature $1/2\rho' = qn/pn^2$, and $1/2\rho = QN/PN^2$, ultimately, so that

$$\frac{1}{\rho'} - \frac{1}{\rho} = -\frac{(2k+1)}{f}.$$

This gives the relation between the curvatures of an object and its image.

The curvature of the image is independent of the position of the

object; for the relation between the curvatures of the object and its image is independent of the distances u, v.

If the object be plane, so that the curvature of the section of the object is zero, $1/\rho = 0$, and therefore

$$\frac{1}{\rho'} = -\frac{(2k+1)}{f}.$$

The radius of curvature has therefore a sign opposite to that of f.

The value of k depends, as has been shown, on the focus chosen to represent the image; for a primary focus, $k = 1 + 1/2\mu$; for a secondary focus, $k = 1/2\mu$, and for the circle of least confusion k is the mean of these values.

For the geometrical focus, $k = 0$, and therefore

$$\frac{1}{\rho'} - \frac{1}{\rho} = -\frac{1}{f}.$$

161. The characteristic function may be applied to investigate the theory of any optical instrument symmetrical about an axis.

For, take two fixed points O, O' on the axis, in the first and final media, respectively, as origins, the axis of the telescope being the axis of z, the positive direction of the axis at O being opposite to the direction of the incident light, and the positive direction of that at O' being in the same as that of the emergent light. Consider the part of the characteristic function between two points (x, y, z), (x', y', z') on any ray, situated respectively in the first and final media. Let the ray through these points meet the planes of xy at O, O' in the points (ξ, η), (ξ', η'). Then if V be the value of the characteristic function from the point (ξ, η) to the point (ξ', η'), and V_0 the value from O to O', V may be expanded by Taylor's Theorem in the form,

$$\begin{aligned} V = V_0 &+ \xi\frac{dV}{d\xi} + \eta\frac{dV}{d\eta} + \xi'\frac{dV}{d\xi'} + \eta'\frac{dV}{d\eta'} \\ &+ \tfrac{1}{2}\xi^2\frac{d^2V}{d\xi^2} + \xi\eta\frac{d^2V}{d\xi d\eta} + \tfrac{1}{2}\eta^2\frac{d^2V}{d\eta^2} \\ &+ \xi\xi'\frac{d^2V}{d\xi d\xi'} + \xi\eta'\frac{d^2V}{d\xi d\eta'} + \xi'\eta\frac{d^2V}{d\xi' d\eta} + \eta\eta'\frac{d^2V}{d\eta d\eta'} \\ &+ \tfrac{1}{2}\xi'^2\frac{d^2V}{d\xi'^2} + \xi'\eta'\frac{d^2V}{d\xi' d\eta'} + \tfrac{1}{2}\eta'^2\frac{d^2V}{d\eta'^2} \\ &+ \text{terms involving higher powers of } \xi, \eta, \xi', \eta'. \end{aligned}$$

In all the coefficients, it is supposed that ξ and η are made equal to zero after differentiation, so that the coefficients are constants. We may therefore write the value of V in the form

$$\begin{aligned}V = V_0 &+ f\xi + g\eta + f'\xi' + g'\eta' \\ &+ \tfrac{1}{2} a\xi^2 + c\xi\eta + \tfrac{1}{2} b\eta^2 \\ &+ p\xi\xi' + q\xi\eta' + r\xi'\eta + s\eta\eta' \\ &+ \tfrac{1}{2} a'\xi'^2 + c'\xi'\eta' + \tfrac{1}{2} b'\eta'^2 + \ldots\ldots\end{aligned}$$

Let U be the characteristic function from the point (x, y, z) to the point (x', y', z'); then from the definition of this function

$$U = V + \mu\,\{(x-\xi)^2 + (y-\eta)^2 + z^2\}^{\frac{1}{2}} + \mu'\,\{(x'-\xi')^2 + (y'-\eta')^2 + z'^2\}^{\frac{1}{2}},$$

or

$$\begin{aligned}U = V_0 &+ \mu\,\{(x-\xi)^2 + (y-\eta)^2 + z^2\}^{\frac{1}{2}} \\ &+ \mu'\,\{(x'-\xi')^2 + (y'-\eta')^2 + z'^2\}^{\frac{1}{2}} \\ &+ f\xi + g\eta + f'\xi' + g'\eta' \\ &+ \tfrac{1}{2} a\xi^2 + c\xi\eta + \tfrac{1}{2} b\eta^2 \\ &+ p\xi\xi' + q\xi\eta' + r\xi'\eta + s\eta\eta' \\ &+ \tfrac{1}{2} a'\xi'^2 + c'\xi'\eta' + \tfrac{1}{2} b'\eta'^2 + \ldots\ldots\end{aligned}$$

But since the function is symmetrical with regard to the axis of z, we must have

$$\begin{array}{llll} f = 0, & g = 0, & f' = 0, & g' = 0, \\ & c = 0, & c' = 0, & \\ & q = 0, & r = 0, & \\ a = b, & a' = b', & p = s. & \end{array}$$

The characteristic function, therefore, reduces to

$$\begin{aligned}U = V_0 + \mu z + \mu' z' + \frac{\mu}{2z}\left\{(x-\xi)^2 + (y-\eta)^2\right\} \\ + \frac{\mu'}{2z'}\left\{(x'-\xi')^2 + (y'-\eta')^2\right\} + \tfrac{1}{2} a\,(\xi^2 + \eta^2) + \tfrac{1}{2} a'\,(\xi'^2 + \eta'^2) \\ + p\,(\xi\xi' + \eta\eta') + \ldots\ldots,\end{aligned}$$

the square roots being expanded by the Binomial Theorem.

The characteristic function is stationary for small variations of ξ, η, ξ', η', and therefore, rejecting higher powers of small quantities,

$$\left(a+\frac{\mu}{z}\right)\xi+p\xi'=\mu\frac{x}{z},$$

$$\left(a+\frac{\mu}{z}\right)\eta+p\eta'=\mu\frac{y}{z},$$

$$p\xi+\left(a'+\frac{\mu'}{z'}\right)\xi'=\mu'\frac{x'}{z'},$$

$$p\eta+\left(a'+\frac{\mu'}{z'}\right)\eta'=\mu'\frac{y'}{z}.$$

Solving the first and third equations to find ξ, ξ', we get

$$\left.\begin{aligned} D\xi &= \frac{\mu x}{z}\left(a'+\frac{\mu'}{z'}\right)-\frac{\mu' x'}{z'}p \\ D\xi' &= \frac{\mu' x'}{z'}\left(a+\frac{\mu}{z}\right)-\frac{\mu x}{z}p \end{aligned}\right\},$$

where
$$D=\left(a+\frac{\mu}{z}\right)\left(a'+\frac{\mu'}{z'}\right)-p^2,$$

and similar formulæ hold for η, η' in terms of y, y'.

Hence

$$\left.\begin{aligned} D(x-\xi) &= \left\{a\left(a'+\frac{\mu'}{z'}\right)-p^2\right\}x+\frac{\mu' x' p}{z'} \\ D(x'-\xi') &= \left\{a'\left(a+\frac{\mu}{z}\right)-p^2\right\}x'+\frac{\mu x p}{z} \end{aligned}\right\}.$$

But the characteristic function U may be written in the form,

$$U=V_0+\mu z+\mu' z'+\frac{\mu}{2z}(x^2-2x\xi)+\frac{\mu'}{2z'}(x'^2-2x'\xi')$$

$$+\tfrac{1}{2}\xi\left\{\left(a+\frac{\mu}{z}\right)\xi+p\xi'\right\}+\tfrac{1}{2}\xi'\left\{p\xi+\left(a'+\frac{\mu'}{z'}\right)\xi'\right\}$$

$$+\text{ similar terms in } y,\ \eta$$

$$=V_0+\mu z+\mu' z'+\frac{\mu x}{2z}(x-\xi)+\frac{\mu' x'}{2z'}(x'-\xi')$$

$$+\text{ similar terms in } y,\ \eta.$$

Substituting the values of $x-\xi$, $x'-\xi'$ in terms of x, x' this becomes

$$U = V_0 + \mu z + \mu' z'$$

$$+ \frac{1}{2D}\left[\frac{\mu}{z}\left\{a\left(a' + \frac{\mu'}{z'}\right) - p^2\right\}x^2 + a^2\frac{\mu\mu'}{zz'}pxx' + \frac{\mu'}{z'}\left\{a'\left(a + \frac{\mu}{z}\right) - p^2\right\}x'^2\right]$$

$$+ \text{a similar term in } y, \eta.$$

Let
$$g = \frac{\mu a'}{p^2 - aa'}, \qquad g' = \frac{\mu' a}{p^2 - aa'},$$

$$f = -\frac{\mu p}{p^2 - aa'}, \qquad f' = -\frac{\mu' p}{p^2 - aa'},$$

now that these letters have ceased to enter into the investigation with other meanings.

Then, if in the expression for U, we multiply the numerator and denominator of the fraction by zz', and divide them by $aa' - p^2$, it easily reduces to

$$U = V_0 + \mu z + \mu' z' + \tfrac{1}{2}\frac{\mu(z' - g')x^2 + \mu'(z - g)x'^2 + (f\mu' + f'\mu)xx'}{(z - g)(z' - g') - ff'}$$

$$+ \tfrac{1}{2}\frac{\mu(z' - g')y^2 + \mu'(z - g)y'^2 + (f\mu' + f'\mu)yy'}{(z - g)(z' - g') - ff'}.$$

162. If (l, m, n) be the direction cosines of the incident ray, through the point (x, y, z), by the properties of the characteristic function we learn that

$$l : m : n = \frac{dV}{dx} : \frac{dV}{dy} : \frac{dV}{dz}.$$

If we put, for brevity

$$\left.\begin{aligned} z - g &= u \\ z' - g' &= u' \end{aligned}\right\}$$

and remember that $f\mu' = f'\mu$, these ratios become

$$\frac{l}{u'x + f'x'} = \frac{m}{u'y + f'y'} = \frac{n}{uu' - ff'}.$$

Similarly if (l', m', n') be the direction cosines of the emergent ray, we may show that

$$\frac{l'}{ux' + fx} = \frac{m'}{uy' + fy} = \frac{n'}{uu' - ff'}.$$

Now suppose that one of the two points, say (x, y, z), remains fixed while the direction of the ray through it changes; then the

first set of these equations determine where any ray meets the plane $z = z'$.

But if x', y', z', be chosen so that the denominators of the ratios vanish, that is, if

$$\left.\begin{aligned} u'x &= -f'x' \\ u'y &= -f'y' \\ uu' &= \quad ff' \end{aligned}\right\},$$

then this set of equations will be satisfied for all values of the ratios $l : m : n$. In other words, whatever be the direction of the incident ray through (x, y, z), it will pass through the point (x', y', z'), as determined by these equations. It is easy to see from the equations giving the ratios $l' : m' : n'$ that this property is reciprocal; hence (x, y, z), (x', y', z') are conjugate points. Their coordinates are connected by the equations

$$uu' = ff',$$

$$\left.\begin{aligned} \frac{x}{x'} = \frac{y}{y'} = -\frac{u}{f} \\ \frac{x'}{x} = \frac{y'}{y} = -\frac{u'}{f'} \end{aligned}\right\}.$$

These equations at once give the positions and characteristic properties of the cardinal points of the system of which we have given an account in connection with Gauss' theory of any system of lenses arranged symmetrically along an axis. Thus if we make u infinite, we get $u' = 0$, and *vice versa;* so that rays which are parallel in the first medium all pass through a point on the plane $u' = 0$, in the second medium; in other words $u' = 0$ is the equation of the second focal plane. Similarly $u = 0$ is the equation of the first focal plane.

Also if we take $u = -f$, and therefore $u' = -f'$, we find

$$\left.\begin{aligned} x &= x' \\ y &= y' \end{aligned}\right\}.$$

This proves that any ray meets the planes $u = -f$, $u' = -f'$, in points, such that the line joining them is parallel to the axis of the system. These are therefore the planes of unit magnification, or principal planes.

From these cardinal points we may deduce the positions of the nodal points, and apply any of the constructions previously

given for determining the position of a focus conjugate to a given focus, or the direction of an emergent ray corresponding to any incident ray.

163. These constructions fail in the case in which $p^2 = aa'$, for then the values of the coordinates of the cardinal points and the principal focal lengths become infinite. The characteristic function becomes

$$U = V_0 + \mu z + \mu' z' + \frac{1}{2}\frac{ax^2 + 2pxx' + a'x'^2}{\dfrac{az}{\mu} + \dfrac{a'z'}{\mu'} + 1}$$
$$+ \text{ a similar term in } y, y'.$$

If we use this form of the characteristic function, the direction cosines of the incident and emergent rays are given by the equations

$$\frac{l}{ax + px'} = \frac{m}{ay + pq'} = \frac{n}{\mu\left(\dfrac{az}{\mu} + \dfrac{a'z'}{\mu'} + 1\right)},$$

$$\frac{l'}{px + a'x'} = \frac{m'}{py + a'y'} = \frac{n}{\mu'\left(\dfrac{az}{\mu} + \dfrac{a'z'}{\mu'} + 1\right)}.$$

By the same reasoning as before, it may be shown that the relations between a pair of conjugate points are determined by the equations

$$\left.\begin{aligned} ax + px' &= 0 \\ ay + py' &= 0 \\ \frac{az}{\mu} + \frac{a'z'}{\mu'} + 1 &= 0 \end{aligned}\right\}.$$

The first two of these equations may be expressed in the forms

$$\left.\begin{aligned} \frac{x}{x'} = \frac{y}{y'} &= -\frac{p}{a} \\ \frac{x'}{x} = \frac{y'}{y} &= -\frac{p}{a'} \end{aligned}\right\}.$$

It appears, therefore, that *the linear magnification is constant for all positions of the object*, namely,

$$\frac{x'}{x} = -\frac{p}{a'} = -k, \text{ say.}$$

Also, for any rays whatever, we have shown that

$$\frac{\mu l}{ax + px'} = \frac{\mu' l'}{px + a'x'},$$

and by virtue of the relation $p^2 = aa'$, this may be written

$$\frac{\mu l}{a} = \frac{\mu' l'}{p}.$$

We may also prove a similar equation between m and m', and therefore the direction cosines of the incident and emergent rays are connected by the relations

$$\frac{l'}{l} = \frac{m'}{m} = \frac{\mu}{\mu'}\frac{p}{a}$$

$$= \frac{\mu k}{\mu'}.$$

The interpretation of this equation is that $\mu k/\mu'$ is the angular magnification; and therefore *the angular magnification is constant.* Rays which are parallel in the first medium are parallel also in the final medium. This is the case of an astronomical telescope directed towards a star when focussed to suit a normally sighted person.

164. *To find the characteristic function for a small pencil after passing through any heterogeneous medium.*

We shall suppose the initial and final media to be homogeneous, of refractive indices μ and μ', respectively. Let O be a fixed point on the principal ray in the first medium and O' a fixed point on the same ray in the final medium; we shall take O, O' for origins and the directions of the principal ray for axes of z in the two media, the positive direction of the axis of z in the first medium being opposite to the direction in which light is travelling, and in the second medium being in the same direction as that of the light. Let any ray meet the plane of xy at O in a point (x, y), and that at O' in a point (x', y'). Then if U be the value of the characteristic function from the point (x, y) to the point (x', y'), and U_0 the value from O to O', U may be expanded by Taylor's theorem in the form

$$U = U_0 + x\frac{dU}{dx} + y\frac{dU}{dy} + x'\frac{dU}{dx'} + y'\frac{dU}{dy'}$$
$$+\tfrac{1}{2}x^2\frac{d^2U}{dx^2} + xy\frac{d^2U}{dxdy} + \tfrac{1}{2}y^2\frac{d^2U}{dy^2}$$
$$+xx'\frac{d^2U}{dxdx'} + xy'\frac{d^2U}{dxdy'} + x'y\frac{d^2U}{dx'dy} + yy'\frac{d^2U}{dydy'}$$
$$+\tfrac{1}{2}x'^2\frac{d^2U}{dx'^2} + x'y'\frac{d^2U}{dx'dy'} + \tfrac{1}{2}y'^2\frac{d^2U}{dy'^2}$$

+ terms involving higher powers of x, y, x', y'.

This may be written

$$U = U_0 + fx + gy + f'x' + g'y'$$
$$+\tfrac{1}{2}ax^2 + cxy + \tfrac{1}{2}by^2$$
$$+pxx' + qxy' + rx'y + syy'$$
$$+\tfrac{1}{2}a'x'^2 + c'x'y' + \tfrac{1}{2}b'y'^2 + \ldots,$$

the constant coefficients being functions depending on the nature of the medium and supposed known.

Let the characteristic function of the incident pencil be

$$V = K + \mu\left(z - \frac{x^2}{2A} - \frac{y^2}{2B} - \frac{xy}{C}\right)$$

and that of the emergent pencil

$$V' = K' + \mu'\left(z' - \frac{x'^2}{2A'} - \frac{y'^2}{2B'} - \frac{x'y'}{C'}\right),$$

then, noting the directions of the axis of z in the two cases, we must have

$$U = V + V';$$

that is,
$$K + K' = U_0 + fx + gy + f'x' + g'y'$$
$$+\tfrac{1}{2}\alpha x^2 + \gamma xy + \tfrac{1}{2}\beta y^2$$
$$+pxx' + qxy' + rx'y + syy'$$
$$+\tfrac{1}{2}\alpha'x'^2 + \gamma'x'y' + \tfrac{1}{2}\beta'y'^2 + \ldots\ldots,$$

where
$$\left.\begin{aligned}\alpha &= a + \frac{\mu}{A}\\ \beta &= b + \frac{\mu}{B}\\ \gamma &= c + \frac{\mu}{C}\end{aligned}\right\},$$

and α', β', γ' denote similar expressions for the second medium.

Now by the properties of the characteristic function this equation must still be true if we let the points (x, y), (x', y') undergo small displacements; by differentiation we arrive at the following equations:

$$\left.\begin{aligned} \alpha x + \gamma y + px' + qy' + f = 0 \\ \gamma x + \beta y + rx' + sy' + g = 0 \\ px + ry + \alpha' x' + \gamma' y' + f' = 0 \\ qx + sy + \gamma' x + \beta' y' + g' = 0 \end{aligned}\right\}.$$

But since x, y, x', y', all vanish together,

$$f = 0, \quad g = 0, \quad f' = 0, \quad g' = 0.$$

Solving the first two equations for x, y in terms of x', y', we find

$$x\delta = (r\gamma - p\beta)\, x' + (s\gamma - q\beta)\, y',$$

$$y\delta = (p\gamma - r\alpha)\, x' + (q\gamma - s\alpha)\, y',$$

where $\delta = \alpha\beta - \gamma^2$.

Hence, substituting these values into the other two equations, we must have

$$x'(\alpha'\delta + pr\gamma - p^2\beta + rp\gamma - r^2\alpha) \\ + y'(\gamma'\delta + ps\gamma - pq\beta + qr\gamma - rs\alpha) = 0,$$

$$x'(\gamma'\delta + qr\gamma - pq\beta + sp\gamma - sr\alpha) \\ + y'(\beta'\delta + qs\gamma - q^2\beta + qs\gamma - s^2\alpha) = 0.$$

These equations must be true for every ray, that is, for all values of x', y', and therefore

$$\left.\begin{aligned} \alpha'\delta = p^2\beta - 2pr\gamma + r^2\alpha \\ \gamma'\delta = pq\beta - (ps + qr)\gamma + rs\alpha \\ \beta'\delta = q^2\beta - 2qs\gamma + s^2\alpha \end{aligned}\right\}.$$

By means of these equations the coefficients α', β', γ' are determined; these serve to determine the constants A', B', C' and all the circumstances of the emergent pencil.

The coordinates (x, y), (x', y') are connected together by a linear transformation. If therefore the small pencil be cut by the plane of xy at O in an ellipse, then the section of the pencil by the plane at O' will also be an ellipse. This proves that if at any point the normal section of the pencil be elliptical, all its normal sections will be elliptical.

EXAMPLES.

1. A ray passes through a right-angled prism with minimum deviation, meeting the face of the prism at a distance y from the edge. Show that the foci of the emergent pencil will be separated by a distance $y\sqrt{2}\,(\mu^2-1)/\mu$.

2. The image of a straight line perpendicular to the axis of a convex lens at a very great distance from it approximates to a parabolic curve, whose equation is

$$2(x+f)+\left(2+\frac{1}{\mu}\right)\frac{y^2}{f}=0,$$

the centre of the lens being the origin, and the circle of least confusion being taken to be the image of any point.

3. A small pencil falls obliquely on a looking-glass, and emerges after reflexion at the silvered back. Show that it proceeds from two focal lines, whose distance from one another is

$$\frac{2t}{\mu}\,\frac{\cos^2\phi'-\cos^2\phi}{\cos^3\phi'},$$

where ϕ and ϕ' are the angles of incidence and refraction at the front of the glass, and t the thickness of the glass.

4. A double convex lens has faces whose radii are each 8 inches, and its refractive index is $(\sqrt{3}+1)/\sqrt{2}$; find its power when its axis is inclined at an angle of 30^0 to the line of sight.

5. A pencil of rays diverging from a point P, whose position is variable, is incident on a refracting sphere at a given point in a given direction; if Q be the corresponding primary focus after refraction through the sphere, G the position of Q when the incident pencil consists of parallel rays, F that of P when Q is at an infinite distance, prove that

$$PF\,.\,QG=\frac{a^2}{16}\,\frac{\sin^2 2\phi\cos^2\phi'}{\sin^2(\phi-\phi')}$$

where a is the radius of the sphere, ϕ the angle of incidence on the sphere and ϕ' the angle of refraction.

Show how to find the corresponding theorem for a pencil refracted through any number of spheres, the axis of the pencil lying always in one plane.

6. A small pencil of parallel rays is refracted centrically through a double convex lens, the radii of whose surfaces are each equal to r, and whose thickness is t; show that, if the square of t be neglected, the distance of the primary focus from the point of emergence of the pencil will be

$$\frac{r\sin\phi'\cos^2\phi}{2\sin(\phi-\phi')}-\frac{t\sin\phi'\cos^2\phi}{4\sin\phi\cos^3\phi'},$$

ϕ and ϕ' being the angles of incidence and refraction.

7. If u be the distance of the origin of light, u' that of the primary focal line from the point of incidence of the axis of a pencil of rays falling obliquely on a refracting sphere, ϕ, ϕ' the angles of incidence and refraction, μ the refractive index, and θ the angular aperture of the small portion of the sphere which refracts the light, prove that, neglecting θ^2 and higher powers

$$\left(\cos\phi+\frac{r\cos^2\phi}{u}\right)\left(1-\frac{3r\theta\sin\phi}{2u}\right)=\mu\left(\cos\phi'+\frac{r\cos^2\phi'}{u'}\right)\left(1-\frac{3r\theta\sin\phi'}{2u'}\right),$$

r being the radius of the sphere.

8. A ray of light $QAST$ passes through a plate of glass bounded by parallel planes, A being the point of incidence and S the point of emergence; prove that if q_1, q_2 be the primary and secondary foci, and $Sq_1=v_1$ $Sq_2=v_2$, $AQ=u$, then

$$v_1=u+\frac{t\cos^2\phi}{\mu\cos^2\phi'},$$

$$v_2=u+\frac{t}{\mu\cos\phi'},$$

where t is the thickness of the plate, μ the refractive index, ϕ the angle of incidence and ϕ' the angle of refraction.

9. A pencil of light is incident obliquely upon a prism, the axis of the pencil lying in a principal plane. If the faces of the prism be slightly curved, and if u, v be the distances of the luminous point and the primary focal line from the point of incidence, ϕ and ϕ' the angles of incidence and refraction of the axis of the pencil which passes through the prism with minimum deviation, then if the length of the path of the pencil inside the prism be so small as to be negligible compared with the other lengths involved, show that

$$\frac{1}{u}-\frac{1}{v}=\frac{1}{\cos^2\phi}(\mu\cos\phi'-\cos\phi)(\sigma-\sigma')$$

when σ, σ' are the curvatures of the faces of the prism.

10. Each of the faces of a thin lens (either double convex or double concave) is a surface whose principal curvatures are ρ and σ, and the directions of principal curvature of one face are at right angles to those of the other. Show that the lens refracts a directly incident small pencil in the same way as a corresponding lens with spherical faces, radii $2(\rho+\sigma)^{-1}$.

11. A person having a very small fragment of a concave reflector, the principal radii of curvature of it being ρ, ρ', wishes to place it so that it may be considered part of a paraboloid of revolution, the positions of the focus and axis being given. Show that it must be placed with its plane of least curvature passing through the focus at the distance $\frac{1}{2}\sqrt{(\rho\rho')}$ therefrom, and with its tangent plane inclined to the axis of the paraboloid and to its focal distance, at an angle whose sine is $\sqrt{(\rho'/\rho)}$.

12. A pencil of light diverging from a point is reflected at any surface; determine the positions and planes of the focal lines of the emergent pencil.

Show that if the reflected pencil converge to one point, the plane of reflexion must be a principal plane of curvature of the surface and that the angle of incidence is given by the equation $\cos\phi=\sqrt{(\rho_1/\rho_2)}$, where ρ_1, ρ_2 are the principal radii of curvature.

13. The front surface of a thin lens is cylindrical (radius ρ), and the back surface is cylindrical (radius ρ'), the axes of the two cylinders being inclined at an angle a, and each perpendicular to the axis of the lens. A pencil from a very distant point passes directly through the lens. Show that the distances of the foci of the emergent pencil (u, v) from the lens are given by

$$\frac{1}{u}+\frac{1}{v}=(\mu-1)\left[\frac{1}{\rho}-\frac{1}{\rho'}\right],$$

$$\frac{1}{u}-\frac{1}{v}=(\mu-1)\sqrt{\left[\frac{1}{\rho^2}-\frac{2\cos 2a}{\rho\rho'}+\frac{1}{\rho'^2}\right]},$$

and find the positions of the principal planes of the emergent pencil.

14. A lens is formed by the intersection of portions of two circular cylinders with their axes at right angles and their concavities inwards. QOQ' is a line at right angles to the axes of the cylinders, and Q a point from which a small pencil falls directly on the lens. Show that in general there will be two distorted images of Q formed by the lens, but that if the radii of the two surfaces are equal, these two images coincide at a distance v from the lens, where

$$\frac{1}{v}-\frac{1}{u}=\frac{1-\mu}{r},$$

r being the common radius and u the distance of Q from the lens.

15. A narrow beam of light is refracted at the surface of a sphere of radius R from a medium whose index is μ to a medium whose index is μ'; prove that, if the plane of incidence be a principal plane of the beam, r, r' the distances from the refracting surface of those focal lines in the two media that are perpendicular to the plane of incidence, and ρ, ρ' the radii of curvature of the sections of the caustic surfaces at these focal lines by the plane of incidence, then these pairs of quantities are connected by the relations

$$\frac{\mu}{r}\cos^2\theta-\frac{\mu}{R}\cos\theta=\frac{\mu'}{r'}\cos^2\theta'-\frac{\mu'}{R}\cos\theta',$$

$$\frac{\mu\rho}{3r^3}\cos^3\theta+\frac{\mu}{rR}\cos\theta\sin\theta-\frac{\mu}{r^2}\cos^2\theta\sin\theta=\frac{\mu'\rho'}{3r'^3}\cos^3\theta'+\frac{\mu'}{r'R}\cos\theta'\sin\theta'-\frac{\mu'}{r'^2}\cos^2\theta'\sin\theta'$$

where θ, θ' are the angles of incidence and refraction of the axis of the beam, the distances being all measured positive in the same direction.

16. Rays of light are incident from the centre upon a reflecting ellipsoid at points situated on a central circular section; prove that the reflected rays pass through the diameter of the ellipsoid perpendicular to this section.

17. Two conical shells of light given by the equation $z^2 = \pm xy$ are incident on the reflecting surface $xyz = a^3$; show that the reflected rays pass through two straight lines at right angles to each other.

18. Show that if the normals to a surface all pass through a given curve, one set of lines of curvature are circles, and those normals which pass through a given point on the curve are generators of a right cone whose axis is the tangent at that point. Hence show that if the normals all pass through two curves, these curves must be conics in planes at right angles to each other, the foci of one being the vertices of the other. (Maxwell.)

19. A mirror in the form of a right circular cylinder stands upon a table so that its axis is normal to the table. Curves traced on the table are viewed by an eye after reflexion in the mirror. Show that the curve on the table which will be seen along a generating line of the cylinder is a straight line, and the curve which will be seen along a circular section of the cylinder is an epitrochoid in which the fixed and rolling circles are equal.

20. A straight line is parallel to the edge of a prism whose vertical angle is ι. Show that when viewed by an eye situated on a line symmetrically placed on the other side of the edge it will appear to lie on a cone of the second order whose equation can be presented in the form

$$x^2 \operatorname{cosec}^2 \tfrac{1}{2}\iota + z^2 = \mu^2 (x^2 + y^2 + z^2).$$

21. A bright line is placed parallel to the axis of a polished cylinder. Show that the curve which is seen on the cylinder by an eye placed in the plane of xy at the point (α, β) lies on the surface

$$b^2y^2 \{(\alpha - x)^2 + (\beta - y)^2 + z^2\} = (\beta x - \alpha y)^2 \{(b - x)^2 + y^2\} + b^2y^2z^2,$$

b being the distance between the bright line and the axis of the cylinder, which is the axis of z.

22. A narrow flat polished piece of watch-spring is bent into the form of a circle, so as to form a cylinder of indefinitely small height; a luminous point is placed so that its projection on the plane of the ring falls within the ring. Show that all the rays after reflexion pass through a straight line of finite length. Calculate the length and position of this line, and point out the modification of the problem when the projection does not fall within the ring.

If a plane surface be placed parallel to the plane of the ring and below it, the bright curve on the plane surface has for its equation $r = a + b\cos\theta$, the point where the line passing through the luminous point and the centre of the ring meets the plane being taken as pole. Trace the curve and find the condition that it may have a cusp.

23. A transparent hollow cylinder stands on a table, and on the inside of the cylinder is wrapped a narrow band of bright reflecting foil in the form of a helix which just makes one revolution. In the centre of the upper rim of the cylinder is placed a luminous point; find the equation of the curve of light on the table, and trace the curve. Prove that the illumination by reflected light at a distance r from the axis of the cylinder varies as

$$\frac{2a \pm r}{r} \left\{4h^2 + (2a \pm r)^2\right\}^{-\frac{3}{2}}$$

where $2h$ is the height, and a the radius of the base of the cylinder.

CHAPTER IX.

DISPERSION AND ACHROMATISM.

165. HITHERTO we have considered light to be simple or homogeneous. The light of the sun, however, is not homogeneous but compound; each ray of solar light is composed of an infinite number of rays of homogeneous light differing from each other in colour and refrangibility. This fact was first established by Newton.

In Newton's experiment his room was darkened and a beam of the sun's light admitted through a small circular hole in the shutter of one of the windows. This beam of light made a small circular spot of white light on the opposite wall. He then placed a triangular prism of glass near the hole, with its edge downwards and perpendicular to the beam of sun-light, so that the rays passed through the prism close to its edge. The patch of light on the wall was no longer circular and white, but elongated and coloured with vivid and intense colours. The sides of the coloured image or *spectrum* were each straight and perpendicular to the edge of the prism, and the ends appeared semi-circular. The breadth of the spectrum was the same as that of the circular white spot, while its length was about five times greater.

This elongation of the image can only be explained by supposing that the rays of the beam of sunlight are refrangible in different degrees. The rays from the sun are not quite parallel, for some might proceed from the upper and others from the lower limb of the sun's disc. But when the prism is placed in its position of minimum deviation, a small difference of incidence will produce no appreciable difference of deviation; consequently the inclination of the emergent rays will be the same as those of the incident rays; and therefore if the beam of light were

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homogeneous it would cause a circular spot of white light of the same dimensions as before, but in a displaced position.

This experiment further shows that those rays which differ in refrangibility differ also in colour; for the coloured spectrum is red at its lower or least refracted end, and the colour changes by imperceptible gradations through yellow, green, blue, until at the upper or most refracted end, it is violet. Newton distinguished seven principal colours; these arranged in order of their refrangibility are red, orange, yellow, green, blue, indigo, violet. Of these the orange and yellow are the most luminous, the red and green next in order, and the indigo and violet weakest.

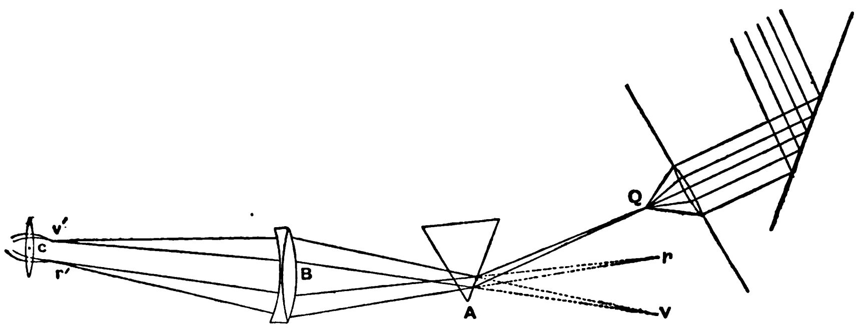
166. After trying several ways of explaining those phenomena Newton was finally led to the following *experimentum crucis*, which is described almost in Newton's own words. He took two boards and placed one of them close behind the prism at the window, so that the light might pass through a small hole, made in it for the purpose, and fall on the other board, which was placed at about twelve feet distance, having first made a small hole in it also for some of that incident light to pass through. Then he placed another prism behind this second board, so that the light passing through the two boards might pass through that also, and be again refracted before it reached the wall. This done, he took the first prism and turned it slowly to and fro about its axis, so as to make the several parts of the image cast on the second board successively pass through the hole in it, and observed to what places on the wall they were refracted by the second prism. He saw that the light tending towards the violet end of the spectrum was considerably more refracted than the light tending towards the red end. Hence he concluded that sun-light is not homogeneous, but consists of rays of different colours, some of which are more refrangible than others.

167. In this form of the experiment the different coloured images of the sun are of considerable size, and are arranged with their centres along a straight line. The coloured images will therefore overlap, and the colours will not be thoroughly separated; the spectrum is then said to be impure. We shall now show how a pure spectrum may be obtained.

The sun is always moving relatively to the earth and therefore

the direction of his rays is continually changing. This change of direction may be corrected by an instrument called a *heliostat*, which consists of a mirror turned by clockwork in such a way that the light is always reflected in the same direction. The reflected rays of the sun are allowed to fall on a convex lens of short focal length, so as to make a very small image of the sun at the focus of the lens; this image may easily be made so small that it may be regarded as a point. A small pencil may be selected from the rays passing through this point by making them fall on a very narrow slit between two carefully worked plates of metal. If a cylindrical lens with its generating lines parallel to the slit be used, the rays may be concentrated on the slit throughout its whole length, and a very bright thin pencil can be obtained. The pencil of light is allowed to fall on a prism near the refracting edge, this edge being parallel to the slit. The prism must be placed in the position of minimum deviation for rays of mean refrangibility, and then it will be nearly in a position of minimum deviation for all rays. The object of this is to make the mean emergent rays diverge from points and not from pairs of focal lines.

Let Q be the small focus or the section of the slit through which the rays pass. Then after refraction at the prism the red



rays will diverge from a point r, and the violet rays from a point v, where $Av = Ar = AQ$. If the colours be received on a screen, they will overlap, and though by moving the screen farther away from the edge of the prism, the colours become more and more separated, yet they become fainter at the same time. The pencil is therefore made to pass through an achromatic lens (the construction of which will be hereafter described), whose centre is B, after which the red rays will converge to a focus r', and the violet rays

to a focus v', where rBr', vBv' are straight lines. The colours are now perfectly separated, but the spectrum $v'r'$ is very small, so that it needs to be magnified before it can be accurately measured. The spectrum is therefore viewed through another lens or eye-piece (also corrected for chromatic dispersion). The two lenses constitute an ordinary astronomical telescope. If therefore the rays from the prism be received on a telescope, by focusing the telescope we shall be able to see a pure spectrum.

If we wish to exhibit the spectrum on a screen, the lens must be removed. In this case it is better to put between Q and A a lens whose focus is at Q. Then the rays after passing this lens are parallel and the points v and r are at an infinite distance; and by moving the screen further from A we separate the colours more and more without weakening their intensity.

168. If a pure solar spectrum be examined carefully, it is found that it is not, as Newton supposed, a continuous coloured band, but that there are at certain intervals abrupt deficiencies of light, forming dark lines across the spectrum. These lines are always seen irregularly disposed along the spectrum whatever refracting substance may be used. When the refracting substance is varied, the positions of the lines change, but they and the coloured rays always appear in the same order, so that any line can be recognised. As these lines are sharp and definite and are always present, they can be used as marks for determining refractive indices; the refractive indices of the rays to which they correspond can be determined for any substance with an accuracy equal to that of astronomical measurements. The positions of these lines, to the number of seven hundred, have been carefully measured and mapped out by Fraunhofer and others, and the refractive indices of the corresponding rays accurately determined for a very large number of substances. By using prisms of the same substance but of different refracting angles, Fraunhofer verified the law of refraction for the rays corresponding to any one of the fixed lines, with extreme accuracy. These dark lines are not characteristic of light in general, but only of solar light; for if the slit be illuminated by a gas-flame, a perfectly continuous spectrum is observed.

The brightness of the solar spectrum is by no means uniform;

it is brightest in the yellow and the neighbouring colours, orange and light green, and falls off gradually on both sides. It may be observed here, though this scarcely belongs to the province of optics, that the solar rays as separated into a spectrum differ from each other also in heating and chemical effects. The heating effect increases as we pass from the violet to the red rays, and still continues to increase for a certain distance beyond the visible spectrum, at the red end. Similarly, if the action of the different rays on a sheet of sensitive paper be observed, the action is very feeble in the red, strong in the blue and violet, and is sensible to a great distance beyond the violet end of the spectrum.

169. There are three different kinds of spectra depending upon the nature of the source of the light employed.

i. The solar spectrum is a continuous spectrum except that it is interrupted by a definite system of dark lines. The spectra of fixed stars also contain dark lines, different for different stars.

ii. The spectra afforded by incandescent solids and liquids are continuous, containing light of all refrangibilities from the extreme red to a higher limit depending on the temperature.

iii. Flames not containing solid particles in suspension, but emitting the light of incandescent gases, give discontinuous spectra, consisting of a definite number of bright lines.

170. Modern experiments have proved that the missing rays in the solar and similar spectra have been removed by absorption. For according to the theory of exchanges it is known that every substance which emits certain kinds of rays to the exclusion of others, absorbs the same kind as it emits; and when the temperatures are the same in the two cases, the amount emitted and the amount absorbed are equal. When an incandescent vapour emitting only rays of certain definite refrangibilities is interposed between the observer and a very bright source of light giving a continuous spectrum, the gas absorbs from the incident light just those rays which itself emits, the light emitted by the gas being substituted for the light it absorbs. It depends on the relative brightness of the two sources whether these particular rays be in excess or defect. If the two sources

be at all comparable in brightness, the rays will be greatly in excess and will appear as bright lines across the spectrum; for these rays constitute the whole light of the one, but only a very small fraction of the light from the other source. But if the brilliancy of the gas be diminished, while that of the source of the continuous spectrum be increased sufficiently, the rays emitted by the gas become less intense than those which have been absorbed, and so by contrast the corresponding lines of the spectrum appear dark. The dark lines in the solar spectrum would therefore be accounted for by supposing that the principal portion of the sun's light comes from an inner mass which gives a continuous spectrum, and that a stratum external to this, contains vapours which absorb particular rays and thus produce dark lines.

For further details connected with the subject of Spectrum Analysis we refer to works dealing specially with that subject.

171. When a ray of light from the sun falls on a prism of glass, we have seen that it is separated into rays of different colours; this fact is called *dispersion*. We shall now seek a proper measure of the *dispersive power* of a substance.

We must first select some ray of the spectrum as a standard ray; we might with advantage choose the ray corresponding to some well-defined dark line occurring about the middle of the spectrum. Let μ be the refractive index of the standard ray.

The measure of the dispersive power of a substance must be independent of the refracting angle of the prism which is used in the experiment. Take a prism of small refracting angle ι, and let D be the deviation for the standard ray; then

$$D=(\mu-1)\,\iota,$$

when the light passes through the prism in a direction nearly perpendicular to its faces. If μ', D' correspond to any other ray of the spectrum, we shall have

$$D'=(\mu'-1)\,\iota;$$

and therefore by subtraction, $\partial D=\iota\,\partial\mu$.

To eliminate ι, we divide this by the former result, so that we get

$$\frac{\partial D}{D}=\frac{\partial\mu}{\mu-1}.$$

This is taken as the measure of the dispersive power of the substance for the ray whose refractive index is $(\mu + \partial\mu)$, and is often denoted by ϖ; thus

$$\varpi = \frac{\partial\mu}{\mu - 1}.$$

172. We may next choose a standard substance. Herschel proposed that water at its temperature of maximum density should be used as a standard, so that any ray might be identified by its refractive index referred to water, or as we might say, by its position on the water scale.

If x be the refractive index of the standard ray referred to this standard substance, and μ the refractive index of the same ray referred to any other substance, then μ will be some function of x, whose form will depend on the nature of the medium. We shall have, therefore,

$$\mu = f(x);$$

and if $x + \partial x$, $\mu + \partial\mu$ are the refractive indices corresponding to any other ray,

$$\begin{aligned} \mu + \partial\mu &= f(x + \partial x) \\ &= f(x) + A\partial x + B(\partial x)^2 + C(\partial x)^3 + \dots \end{aligned}$$

where A, B, C,... are functions of x, independent of the increment ∂x.

Thus $$\partial\mu = A\partial x + B(\partial x)^2 + C(\partial x)^3 + \dots;$$

and since x and μ are constants, we may write this equation in the form

$$\frac{\partial\mu}{\mu - 1} = a\frac{\partial x}{x - 1} + b\left(\frac{\partial x}{x - 1}\right)^2 + c\left(\frac{\partial x}{x - 1}\right)^3 + \dots$$

where a, b, c,..., are constants depending on the medium.

In order to determine the values of the constants a, b, c,..., for each different medium, we must know several corresponding values of the quantities $\partial\mu/(\mu - 1)$, and $\partial x/(x - 1)$. Fraunhofer's observations on the refractive indices of rays corresponding to the fixed lines of the spectrum furnish all the required data. Since the fraction $\partial x/(x - 1)$ is always small, and the constants b, c... are not found to be very great, we may without appreciable error neglect all the terms beyond the first two. The first term is much the most important, and we see that the ratio of

the dispersive powers is nearly constant and equal to a; so that a may be called the dispersive power of the substance in terms of the standard, for all rays. This ratio is not, however, quite constant, and this fact is called the *irrationality of dispersion*. If two prisms be constructed, one of the standard substance and the other of the substance under consideration, then if the spectrum given by each be examined, the fixed lines and coloured rays will occur in the same order in each, but since the ratios of the dispersion of corresponding rays for the two substances are not proportional, the spectra will not be geometrically similar. If the prisms be arranged side by side so as to give spectra of equal lengths and so that the extreme rays in each may correspond in position, the intermediate rays will not exactly correspond in position.

173. We shall now proceed to find the dispersion of a ray of solar light, produced by refraction through a prism.

To make the problem more general as well as more symmetrical, we shall suppose that the light is dispersed before entering the prism. Let ϕ, ϕ' be the angles of incidence and refraction at the first surface of the prism, and ψ', ψ the angles of incidence and emergence at the second surface of the prism, of the standard ray whose refractive index is μ. Then, denoting the angle of the prism by ι, these quantities are connected by the equations

$$\left.\begin{aligned} \sin\phi &= \mu\sin\phi' \\ \sin\psi &= \mu\sin\psi' \\ \phi' + \psi' &= \iota \end{aligned}\right\}.$$

Let $\phi+\partial\phi$, $\phi'+\partial\phi'$, $\psi'+\partial\psi'$, $\psi+\partial\psi$ denote the corresponding angles for any other ray, whose refractive index is $\mu+\partial\mu$. Then by differentiating the relations between the quantities ϕ, ϕ', ψ, ψ', μ, we have

$$\left.\begin{aligned} \cos\phi\,\partial\phi &= \partial\mu\sin\phi' + \mu\cos\phi'\,\partial\phi' \\ \cos\psi\,\partial\psi &= \partial\mu\sin\psi' + \mu\cos\psi'\,\partial\psi' \end{aligned}\right\}$$
$$\partial\phi' + \partial\psi' = 0.$$

From these equations we may eliminate $\partial\phi'$ and $\partial\psi'$ by multiplying the first equation by $\cos\psi'$ and the second by $\cos\phi'$ and adding; in this way we get

$$\begin{aligned} \cos\phi\cos\psi'\,\partial\phi + \cos\psi\cos\phi'\,\partial\psi &= \partial\mu\,\{\sin\phi'\cos\psi' + \sin\psi'\cos\phi'\} \\ &= \partial\mu\sin(\phi'+\psi'), \end{aligned}$$

and therefore

$$\cos\phi\cos\psi'\,\partial\phi+\cos\psi\cos\phi'\,\partial\psi=\partial\mu\sin\iota,$$

an equation which gives the relation between the dispersions of the incident and emergent rays.

Where the incident ray is not dispersed, we must make $\partial\phi$ vanish, and then the dispersion of the emergent ray is given by the equation

$$\partial\psi=\frac{\partial\mu\sin\iota}{\cos\psi\cos\phi'}.$$

Thus *the dispersion varies inversely as the product of the cosines of the angles of refraction at the two surfaces.*

174. If the ray be incident perpendicularly on the first surface $\phi'=0$, $\psi'=\iota$; and therefore, $\sin\psi=\mu\sin\iota$, so that the above result becomes

$$\partial\psi=\frac{\partial\mu}{\mu}\tan\psi.$$

When the deviation is a minimum for the standard ray, $\phi'=\psi'=\frac{1}{2}\iota$ and therefore

$$\sin\iota=2\sin\psi'\cos\phi'=\frac{2}{\mu}\sin\psi\cos\phi',$$

so that

$$\partial\psi=\frac{2\partial\mu}{\mu}\tan\psi.$$

Newton thought that the position of minimum deviation was also the position of minimum dispersion; but the last two results show that this is not the case. To find the position of minimum dispersion, we must make the denominator of the expression for $\partial\psi$ a maximum; that is $\cos\psi\cos\phi'$ must be a maximum. Equating to zero the first differential of this expression, we easily deduce the equation

$$\tan\psi\,\partial\psi+\tan\phi'\,\partial\phi'=0.$$

If in this equation we write $\partial\phi'=-\partial\psi'$, it becomes

$$\tan\psi\,\partial\psi=\tan\phi'\,\partial\psi'.$$

But the angles ψ, ψ' are always connected by the relation $\sin\psi=\mu\sin\psi'$, and if we take logarithms of both sides and then differentiate, we get

$$\cot\psi\,\partial\psi=\cot\psi'\,\partial\psi'.$$

We can now eliminate the ratio $\partial\psi : \partial\psi'$, and we thus arrive at the equation

$$\tan^2\psi = \tan\phi' \tan\psi'.$$

By combining this equation with the equations $\sin\psi = \mu \sin\psi'$ and $\phi' + \psi' = \iota$, the position of minimum dispersion is completely determined.

The elimination of ψ and ϕ' may be conducted in the following manner. The relation between ψ and ψ', when expressed in terms of tangents, becomes

$$\frac{\tan^2\psi}{1+\tan^2\psi} = \frac{\mu^2 \tan^2\psi'}{1+\tan^2\psi'};$$

and by substituting for $\tan^2\psi$ its value $\tan\phi' \tan\psi'$, we get

$$\mu^2 \tan\psi' = \frac{\tan\phi'(1+\tan^2\psi')}{1+\tan\phi'\tan\psi'},$$

or

$$(\mu^2-1)\tan\psi' = \frac{\tan\phi' - \tan\psi'}{1+\tan\phi'\tan\psi'}$$

$$= \tan(\phi' - \psi'),$$

and therefore finally,

$$(\mu^2-1)\tan\psi' = \tan(\iota - 2\psi').$$

This is a cubic equation in $\tan\psi'$, from which ψ' may be determined.

175. If the refracting angle of the prism be less than the critical angle for glass, it is not difficult to see that this minimum always exists. For when the incident ray just grazes the face of the prism, proceeding towards the edge, $\phi = \frac{1}{2}\pi$, and therefore $\phi' = \gamma$, where γ is the critical angle. This is the greatest possible value for ϕ', and therefore ψ' has its least possible value. But if ι be less than γ, ψ' is then negative, and therefore ψ is negative and is numerically greatest. Hence $\cos\psi \cos\phi'$ is a minimum, and therefore the dispersion is a maximum.

As the angle of incidence diminishes, $\cos\phi'$ increases continually until $\phi' = 0$, when $\cos\phi = 1$; after this ϕ' becomes negative, and $\cos\phi'$ diminishes from unity to $\cos(\gamma - \iota)$. Also, $\cos\psi$ increases at first up to unity, when $\psi = 0$, and then ψ becomes positive and increases up to $\frac{1}{2}\pi$, so that $\cos\psi$ diminishes from unity down to

zero. Thus at first the dispersion gets smaller, then it attains a minimum, and afterwards increases without limit.

Since the dispersion may be indefinitely increased by adjusting the position of the prism with respect to the incident ray, it follows that *the dispersion produced by a prism, whose refracting angle is ever so great, may be counteracted by the dispersion of another prism of the same material, whose refracting angle is ever so small.*

176. *To find the dispersion produced by two prisms whose refracting edges are parallel.*

Let ι' be the refracting angle of the second prism, and μ' the refractive index for the standard ray of the substance of which it is composed; and let ϕ'', ϕ''', ψ'', ψ''' be the angles corresponding to ϕ, ϕ', ψ, ψ' of the first prism. Then we have the following relations between the angles

$$\begin{aligned} \sin\phi &= \mu \sin\phi', \qquad & \sin\psi &= \mu\sin\psi', \\ \sin\phi'' &= \mu' \sin\phi''', \qquad & \sin\psi'' &= \mu' \sin\psi''', \\ \phi' + \psi' &= \iota, \qquad & \phi''' + \psi''' &= \iota'. \end{aligned}$$

Also, if the inclinations of the adjacent faces be θ, then

$$\psi + \phi'' = \theta.$$

Taking variations of all these quantities corresponding to a ray whose refractive indices are $\mu + \partial\mu$, $\mu' + \partial\mu'$, we get

$$\left.\begin{aligned} \cos\phi\cos\psi'\,\partial\phi + \cos\psi\cos\phi'\,\partial\psi &= \partial\mu\sin\iota \\ \cos\phi''\cos\psi'''\,\partial\phi'' + \cos\psi''\cos\phi'''\,\partial\psi'' &= \partial\mu'\sin\iota' \end{aligned}\right\},$$

$$\partial\psi + \partial\phi'' = 0.$$

To eliminate $\partial\psi$ and $\partial\phi''$, we have only to substitute their values from the first equations into the last, and the equation thence resulting will contain only $\partial\phi$ and $\partial\psi''$, and it will therefore be a relation between the dispersions of the initial and final rays.

177. We may proceed in the same manner to find the dispersions produced by the combination of any number of prisms whose axes are parallel.

For brevity, let

$$q = \frac{\sin\iota}{\cos\psi\cos\phi'}, \quad p = \frac{\cos\phi}{\cos\phi'}\frac{\cos\psi'}{\cos\psi},$$

where the angles ϕ, ϕ', ψ, ψ', ι belong to any one of the prisms, and suppose that there are n prisms, distinguished by symbols with suffixes 1, 2, 3...n. Then if we apply the equations already obtained to each prism in succession, we obtain the following equations

$$\left.\begin{array}{l} p_n\ \partial\phi_n\ + \partial\psi_n\ = \partial\mu_n\ q_n \\ p_{n-1}\,\partial\phi_{n-1} + \partial\psi_{n-1} = \partial\mu_{n-1}\,q_{n-1} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ p_2\,\partial\phi_2 + \partial\psi_2 = \partial\mu_2\,q_2 \\ p_1\,\partial\phi_1 + \partial\psi_1 = \partial\mu_1\,q_1 \end{array}\right\},$$

$$\left.\begin{array}{l} \partial\psi_{n-1} + \partial\phi_n\ = 0 \\ \partial\psi_{n-2} + \partial\phi_{n-1} = 0 \\ \cdots\cdots\cdots\cdots\cdots \\ \partial\psi_1\ \ + \partial\phi_2\ = 0 \end{array}\right\}.$$

If we multiply the first set of equations by 1, p_n, $p_n p_{n-1}$, $p_n p_{n-1} p_{n-2}$, ..., $p_n p_{n-1}$...... p_2, and add them, all the angles but $\partial\psi_n$ and $\partial\phi_1$ will disappear by virtue of the second set of equations; so that

$$\partial\psi_n + p_n\,p_{n-1}\cdots p_2\,p_1\,\partial\phi_1 = \partial\mu_n\,q_n + \partial\mu_{n-1}\,q_{n-1}\,p_n$$

$$+ \partial\mu_{n-2}\,q_{n-1}\,p_n\,p_{n-1} + \ldots + \partial\mu_1\,q_1\,p_n\,p_{n-1}\cdots p_2.$$

This is the relation between the angles of dispersion of the pencil at incidence and emergence.

178. When the refracting angles of the prisms are small, and the ray passes nearly perpendicularly through them, the value of the dispersion assumes a very simple form. For if ι, ι', ι''... be the refracting angles of the prisms and μ, μ', μ'' ... the refractive indices of the standard ray for the different substances of which they are composed, the deviation produced by the first prism will be

$$D = (\mu - 1)\,\iota;$$

and there will be similar expressions for all the subsequent deviations. If we denote the whole deviation by Δ, we get by addition,

$$\Delta = (\mu - 1)\,\iota + (\mu' - 1)\,\iota' + (\mu'' - 1)\,\iota'' + \ldots$$

and therefore for any other ray the dispersion will be given by the equation

$$\partial\Delta = \partial\mu\,\iota + \partial\mu'\,\iota' + \partial\mu''\,\iota'' + \ldots$$

Let ϖ, ϖ', ϖ'' be the dispersive powers of the different substances for this ray, then the value of the total dispersion becomes

$$\partial\Delta = \varpi\,(\mu - 1)\,\iota + \varpi'\,(\mu' - 1)\,\iota' + \ldots$$

179. If a ray of light be made to pass through two prisms in succession, it is always possible to adjust their refracting angles, so that the dispersion produced by the first may be counteracted approximately by the second, and consequently that the emergent ray may be without colour.

This Newton conceived to be impossible, without at the same time making the deviations of the two prisms counteract one another, so that the whole deviation of the pencil would disappear. He appears to have been misled by an accidental circumstance in an experiment in which he counteracted the refraction of a glass prism by enclosing it in a water prism; he had mixed sugar of lead with the water to increase its refractive power and this gave it a higher dispersive power also, and it so happened that the emergent ray was colourless, when by properly adjusting the angle of the water prism the emergent ray was made parallel to the incident ray. From this he concluded that the dispersion of all substances was proportional to the deviation of the mean ray, and that therefore the dispersion could never be destroyed so long as any refraction took place. This made him despair of improving refracting telescopes, and led him to turn his attention to the application of mirrors to these instruments.

Newton's mistake was first discovered by a gentleman of Worcestershire named Hall, who made the first achromatic telescope. This discovery, however, was allowed to fall into oblivion, until the experiment was again tried by Dollond, an optician in London, who found that the dispersion could be corrected without destroying the refraction, and therefore that Newton's conclusion was not correct.

We have seen however that different coloured rays are not dispersed in the same proportion by different substances; or in other words, that the spectra formed by prisms of different

substances are not geometrically similar. Hence, if the prisms be arranged so as to unite two rays (for example, the extreme red and the extreme violet rays) in the emergent beam, there will be still a small dispersion of the other rays. Thus the beam instead of emerging quite colourless, will form a second but much smaller spectrum; this is called the *secondary spectrum.*

Also, it will be found that by using three prisms of three different materials, three rays of the emergent beam (for example, the red, green and violet) may be united; but still, owing to the irrationality of dispersion, the other rays will not be quite united, and there will be another still smaller spectrum called a tertiary spectrum; and so on indefinitely. In theory, therefore, it is impossible to attain perfect achromatism, without the use of a very large number of different media; yet in practice these successive spectra rapidly grow fainter and become insensible; so much so, that it is seldom deemed necessary to combine more than two rays. The two rays selected will not be the extreme red and violet rays, because these are comparatively faint; it is better to combine the two rays whose brightness and difference of colour are greatest, such as a ray from the yellow-orange and one from the green-blue.

The first successful attempt to get rid of the secondary spectra was made by Blair; an account of his work was published in the *Phil. Trans. Edin.,* 1791. He found that in the spectrum of hydrochloric acid, the more refrangible part of the spectrum, green to violet, was much more contracted, and the less refrangible part of the spectrum more dilated, than in most metallic solutions; and by mixing the chlorides of antimony and of mercury in suitable proportions with hydrochloric acid, or with salammoniac, he obtained a fluid which, while having a different absolute dispersion from crown-glass, gave a spectrum geometrically similar to that of crown-glass. When a combination of two lenses or two prisms was constructed out of this fluid medium and crown-glass, in such a way that in the emergent beam of light two differently coloured rays should be united, the emergent beam was absolutely without colour. Blair's object-glasses were considered as of singular merit at the time, but through certain inconveniences attending lenses made of fluid media they never came into use.

What Blair effected with fluid lenses, Professor Abbé of Jena

claims to have now achieved by his discoveries of new kinds of glass. In 1881, Professor Abbé, assisted by Dr Schott, commenced the work of examining the optical properties of all glasses, that is, of all known substances which undergo vitreous fusion and solidify in non-crystalline transparent masses. The work was continued till the end of 1883, and directed towards the solution of two practical problems. The first of these was the production of pairs of kinds of flint and crown-glass, such that the dispersion in the various regions of the spectrum should be, for each pair, as nearly as possible proportional. The second problem was the production of a greater multiplicity in the gradations of optical glass, in respect of the two chief optical constants, the index of refraction and the mean dispersion. The first problem has been satisfactorily solved, with the result that achromatic lenses of a much more perfect kind than have ever before been attainable are now being manufactured; and the second has also been successfully carried out, and a whole series of new glasses of graduated properties are at the service of the optician.

180. We shall now find the condition that when a ray of light from the sun falls upon a combination of two prisms, the emergent ray may be colourless.

To do this we shall investigate the condition that two of the brightest rays may be united in the emergent beam, and shall suppose that the secondary spectrum is so small that it may be neglected. Let one of the rays be chosen as the standard ray, and let the refractive index of the other ray be expressed by means of a small variation from that of the standard ray, as before. Then, since the incident and emergent rays are united, we shall have $\partial\phi = 0$, $\partial\psi'' = 0$, in the equations of § 176; these equations will therefore be

$$\cos\psi \cos\phi' \,\partial\psi = \partial\mu \sin\iota,$$

$$\cos\phi'' \cos\psi''' \,\partial\phi'' = \partial\mu' \sin\iota',$$

$$\partial\psi + \partial\phi'' = 0.$$

If we eliminate $\partial\psi$ and $\partial\phi''$ from these equations, we find

$$\partial\mu \sin\iota \cos\phi'' \cos\psi''' + \partial\mu' \sin\iota' \cos\psi \cos\phi' = 0.$$

The angles ψ, ϕ', ϕ'', ψ''' which occur in this equation are connected by the relations

$$\sin\psi = \mu\sin(\iota - \phi'),\ \sin\phi'' = \mu'\sin(\iota' - \psi''');$$

so that if μ, μ' be given, there are four independent angles entering into the equation of condition, namely, ι, ι', ψ and ϕ''. Hence if the first prism be given in position so that the light falls upon it at a given angle of incidence, the angles ι and ψ will be given, and ι' and ϕ'' will remain arbitrary. The equation of condition may therefore be satisfied in two ways, either by fixing the position of the prism and varying its refracting angle, or by varying its position when the refracting angle is given. If the prisms are of the same material, so that $\partial\mu = \partial\mu'$, the emergent beam may still be achromatised in either of the two ways.

If the prisms are both placed in their position of minimum deviation for the standard ray, the equation of condition for achromatism assumes a simpler form. For in this case,

$$\phi' = \psi' = \tfrac{1}{2}\iota \text{ and } \phi''' = \psi''' = \tfrac{1}{2}\iota';$$

so that the equation becomes

$$\partial\mu \sin\iota\cos\phi''\cos\tfrac{1}{2}\iota' + \partial\mu'\sin\iota'\cos\phi\cos\tfrac{1}{2}\iota = 0.$$

If now we divide each side by $2\cos\frac{1}{2}\iota\cos\frac{1}{2}\iota'$, and notice that

$$\sin\phi = \mu\sin\tfrac{1}{2}\iota,\ \sin\phi'' = \mu'\sin\tfrac{1}{2}\iota',$$

this equation reduces to

$$\mu'\partial\mu\tan\phi + \mu\partial\mu'\tan\phi'' = 0,$$

where ϕ and ϕ'' are determined by the two preceding equations.

181. When there are n prisms, the condition of achromatism may be found in the same way. To unite two rays, one of which is chosen as the standard ray, and the other a ray whose refractive indices for the different media differ from those of the standard ray by small increments, we must make $\partial\phi_1 = 0$ and $\partial\psi_n = 0$ in the equation of § 177, and then the condition reduces to

$$\partial\mu_n q_n + \partial\mu_{n-1} q_{n-1} p_n + \partial\mu_{n-2} q_{n-2} p_n p_{n-1} + \ldots + \partial\mu_1 q_1 p_n p_{n-1} \cdots p_2 = 0.$$

When the ray of light passes nearly perpendicularly through a series of prisms of small refracting angles, this equation assumes the simple form

$$\partial\mu_1\iota_1 + \partial\mu_2\iota_2 + \ldots + \partial\mu_n\iota_n = 0.$$

With n given prisms, it is possible to form a combination which will unite n rays of the spectrum. For suppose that the substances and the refracting angles of all the prisms are prescribed, and further that the first prism is fixed in position, so that the light falls upon it at a given angle of incidence, then the inclinations of the adjacent faces of consecutive prisms will still be at our disposal. These $(n-1)$ arbitrary angles will enable us to satisfy the $(n-1)$ equations which express the conditions that $(n-1)$ rays of the spectrum may emerge in the same direction as the standard ray. If the prisms be made of the same material, $\partial\mu_n = \partial\mu_{n-1} = \ldots = \partial\mu_1$, and therefore if the combination be achromatic for one pair of colours, all the coloured rays will be united. In this case, perfect achromatism may be secured by one relation among the angles at our disposal.

Achromatism of lenses.

182. By the proper combination of lenses the dispersion of differently coloured lights may approximately be destroyed; just as in the case of light passing through two prisms, the dispersion produced by one lens may be approximately counteracted by that produced by a second lens, so that the emergent rays may be without colour.

We shall first confine our attention to the approximate theory of lenses, in which the thickness of the lens is neglected and the principal points considered as coinciding in one point called the centre of the lens. For the accurate theory of lenses becomes in this case much complicated by the fact that the principal points of the lenses, from which all distances are usually measured, themselves vary in position according to the refractive index of the particular ray we are considering.

In all cases we shall let μ be the refractive index of the standard ray, and $\mu + \partial\mu$ the refractive index of any other ray. The focal lengths of the lenses will be supposed to be expressed in terms of the refractive index of the standard ray.

It will be useful to find the change in the focal length of a lens, as the ray changes from the standard ray, to any other. The

value of the focal length of a double convex lens, the radii of whose bounding surfaces are r, s, respectively, is given by the equation,

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{r} + \frac{1}{s}\right),$$

where μ is the refractive index of the substance for the standard ray. Giving a small variation to μ, so that it becomes $\mu + \partial\mu$, this equation gives

$$\partial\left(\frac{1}{f}\right) = \partial\mu\left(\frac{1}{r} + \frac{1}{s}\right)$$
$$= \frac{\partial\mu}{\mu - 1}\,\frac{1}{f};$$

and therefore, if we denote the dispersive power of the medium by ϖ, the variation of the focal length is determined by the equation

$$\partial\left(\frac{1}{f}\right) = \frac{\varpi}{f}.$$

183. When an image is formed by a lens or system of lenses which is not achromatic, the light being not homogeneous, it will be affected by dispersion in the lenses in two particulars; first, the different coloured images will be distributed in different positions along the axis of the system, and secondly, the coloured images will have different magnitudes. In certain cases both these defects can be removed, in other cases only one of them can be removed, and to choose which correction shall be made, it will be necessary to consider the use to which the system is to be applied, so as to remove the defect which is of most consequence.

For the object-glass of a telescope two lenses are used, and are placed close together so as to act as one lens. Then a point and its image always lie on the same line through the centre of the lens, so that if the lenses be corrected so that the differently coloured images all lie in the same plane perpendicular to the axis, they will all have the same magnitude. It will therefore be necessary only to make the first correction, and then the other will be satisfied.

These object-glasses are usually made of a double convex lens of crown-glass outside, combined with a double concave lens of flint-glass, which has a higher dispersive power than crown-glass. It is easy to see in a general way how the correction may be

effected. By the convex lens the coloured images will be formed at different distances along the axis, the violet image being the nearest to the lens, and the red image the most remote from it. The effect of the concave lens on these images will be to throw them farther away from the lens, and the effect on the violet image will be stronger than that on the red image. By a proper adjustment of the lenses, the final violet image may be made to coincide with the final red image, or any two other colours may be united in the final image. If the lenses were of the same kind of glass, in order that the dispersion produced by the one should be neutralized by that produced by the other, the lenses would have to be such that the deviation produced by the two lenses would also destroy each other, and therefore the combination would not produce an image at all. But it has been seen that for different kinds of glass the dispersion is not proportional to the deviation, but that flint-glass has a higher dispersive power than crown-glass, so that it is possible to destroy the dispersion without destroying the deviation.

184. We shall now investigate the condition that a combination of two lenses made of different kinds of glass, placed close together, may be achromatic for two given colours.

We shall suppose that one of the colours is the standard colour, and that the focal lengths of the two lenses are f, f', respectively. There will be two images; the first being the image of the object formed by the first lens, and the second being the image of this first image formed by the second lens. Let x, x' be the distances of the object and the first image in front of, and behind, the centre of the first lens, y', y the distances of the first and second images in front of, and behind, the centre of the second lens, respectively. Then

$$\frac{1}{x}+\frac{1}{x'}=\frac{1}{f},$$

$$\frac{1}{y}+\frac{1}{y'}=\frac{1}{f'}.$$

If we neglect the thicknesses and the distance between the lenses, $y'=-x'$, and therefore

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{f}+\frac{1}{f'}.$$

The condition that the system should be achromatic is that y should be the same for the two colours; and therefore, since x is independent of the colour,

$$\partial\left(\frac{1}{f}\right)+\partial\left(\frac{1}{f'}\right)=0,$$

that is,

$$\frac{\varpi}{f}+\frac{\varpi'}{f'}=0.$$

This is the condition of achromatism for the combination.

This condition is independent of x and y, so that *the combination will be achromatic for objects at all distances.* It is immaterial in what order the lenses are placed.

In the construction of microscopic object-glasses, achromatic couples of this kind are very generally used, each consisting of a plano-concave lens of flint cemented to a double convex of crown, the plane face being exposed to the incident light.

185. If the combination of two lenses in contact has been over-corrected for dispersion, that is, if the violet image formed by the two lenses be at a greater distance from them than the red image, the defect may be removed by slightly separating the two lenses. The distance between the lenses must be very small, or else when the coloured images are corrected for distance they will not be corrected for magnitude.

The same equations hold good as before, namely,

$$\left.\begin{aligned}\frac{1}{x}+\frac{1}{x'}&=\frac{1}{f}\\ \frac{1}{y}+\frac{1}{y'}&=\frac{1}{f'}\end{aligned}\right\};$$

and besides these there is the equation

$$x'+y'=a,$$

a being small. Then supposing the coloured images to be formed at the same distance, $\partial x=0$, and $\partial y=0$, and therefore

$$\left.\begin{aligned}-\frac{\partial x'}{x'^2}&=\frac{\varpi}{f},\\ -\frac{\partial y'}{y'^2}&=\frac{\varpi'}{f'};\end{aligned}\right\},$$

and since $$\partial x' + \partial y' = 0,$$

$$x'^2 \frac{\varpi}{f} + y'^2 \frac{\varpi'}{f'} = 0.$$

Substituting for y' its value $a - x'$, and neglecting squares of the ratio $a : x'$, this equation reduces to

$$2a \frac{\varpi'}{f'} = x' \left\{ \frac{\varpi}{f} + \frac{\varpi'}{f'} \right\},$$

or
$$2a \frac{\varpi'}{f'} \left\{ \frac{1}{f} - \frac{1}{x} \right\} = \frac{\varpi}{f} + \frac{\varpi'}{f'}.$$

Thus the distance a is not independent of the position of the object; but when the combination is used as the object-glass of a telescope, the distance of the object x is very large compared with the focal lengths. Neglecting, therefore, the reciprocal of x, the equation of condition is

$$2a \frac{\varpi'}{ff'} = \frac{\varpi}{f} + \frac{\varpi'}{f'};$$

or, since $\frac{\varpi'}{f'}$ is very nearly equal to $-\frac{\varpi}{f}$,

$$-\frac{2a\varpi}{f^2} = \frac{\varpi}{f} + \frac{\varpi'}{f'}.$$

This shows that if the correction is to be possible, $\varpi/f + \varpi'/f'$ must be negative.

But if in the first form of the combination, ∂y be the change of y due to the change $\partial\mu$ in the refractive index,

$$-\frac{\partial y}{y^2} = \frac{\varpi}{f} + \frac{\varpi'}{f'},$$

and therefore ∂y must be positive. The violet rays will therefore form an image at a greater distance from the lens than the red rays; that is to say, the original lens was over-corrected.

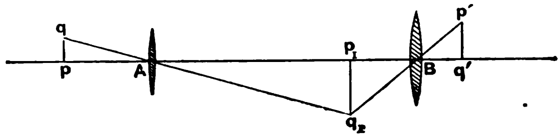
186. If three thin lenses, formed of media of different dispersive powers, be combined into a single lens, the system may be made achromatic to a higher degree of approximation; the coloured images formed by three different kinds of light may be united. More generally, if n lenses form a combination, whose thickness may

be neglected, the system will unite the images formed by rays whose refractive indices are μ and $\mu + \partial\mu$, provided that

$$\Sigma\left(\frac{\varpi}{f}\right) = 0.$$

This may be proved in the same way as before. The equation of condition can be satisfied for $n-1$ systems of values of $\partial\mu$, and therefore the images corresponding to n lines of the spectrum may be united.

187. When the two lenses forming a combination are separated by an interval, it is impossible simultaneously to effect the two corrections for dispersion.



For let x, x' be the distances of the object and its first image in front of, and behind, the first lens, y', y the distances of the first and final image in front of, and behind, the second lens, respectively, and let β, β_1, β' be the linear magnitudes of the object and its images. Then the following ratios must hold:

$$\left.\begin{aligned}\frac{\beta}{\beta_1} &= -\frac{x}{x'}\\ \frac{\beta_1}{\beta'} &= -\frac{y'}{y}\end{aligned}\right\},$$

and therefore
$$\frac{\beta}{\beta'} = \frac{xy'}{x'y}.$$

If the coloured images corresponding to refractive indices μ, $\mu + \partial\mu$ be formed at the same distance and also have the same magnitude,

$$\left.\begin{aligned}\partial x &= 0\\ \partial y &= 0\end{aligned}\right\},$$

and
$$\partial\left(\frac{xy'}{yx'}\right) = 0;$$

and therefore
$$\left(\partial\frac{y'}{x'}\right) = 0.$$

But $x' + y' = a$, where a denotes the distance between the lenses; so that it is necessary that $\partial y'$, $\partial x'$ should both vanish. In other words each lens must be achromatic of itself. This cannot be effected unless each lens of the combination be itself an achromatised couple of lenses in contact.

188. It is often necessary, however, to correct a system of two lenses separated by an interval, for errors due to dispersion, as far as possible; so that we must choose which of the two corrections shall be effected, and which left.

It is then usual to make the coloured images have the same magnitude; for the eye is a better judge of the magnitude of an object than of its distance.

Using the same notation as before, the condition is that

$$\partial\left(\frac{\beta}{\beta'}\right) = 0.$$

But we have seen that

$$\frac{\beta}{\beta'} = \frac{xy'}{yx'}$$

$$= \frac{x}{x'}\left\{\frac{y'}{f'} - 1\right\},$$

by virtue of the equation $\dfrac{1}{y} + \dfrac{1}{y'} = \dfrac{1}{f'}$.

Also $x' + y' = a$; and therefore

$$\frac{\beta}{\beta'} = \frac{x}{x'}\left(\frac{a - x'}{f'} - 1\right)$$

$$= \frac{x}{x'}\left(\frac{a}{f'} - 1\right) - \frac{x}{f'}$$

$$= \left(\frac{x}{f} - 1\right)\left(\frac{a}{f'} - 1\right) - \frac{x}{f'},$$

or finally,

$$\frac{\beta}{\beta'} = 1 - \frac{x}{f} - \frac{x + a}{f'} + \frac{ax}{ff'}.$$

Equating to zero the variation of this expression, we get

$$\frac{x\varpi}{f} + \frac{(x + a)\,\varpi'}{f'} = \frac{ax\,(\varpi + \varpi')}{ff'}.$$

This is therefore the condition for the partial achromatism of the two lenses. In general, it is not independent of the position of the object.

189. If we consider the inclinations of rays to the axis of the instrument, instead of the magnifying power, it will be seen that we have ensured that two coloured rays diverging from the object will emerge parallel to each other.

For if α, α' be the inclinations to the axis of the original and final rays, cutting the axis at the points determined by x, y, we may see directly from a figure, or by Helmholtz' theorem relating to the magnifying power, that

$$\frac{\beta}{\beta'} = \frac{\tan \alpha'}{\tan \alpha} = \frac{xy'}{x'y};$$

so that if the condition previously found be satisfied, then $\partial x' = 0$; and the final rays emerge parallel to each other.

190. The most useful application of this condition is to the achromatism of eye-pieces. The rays strike the eye-piece excentrically diverging from the image formed by the object-glass. The images formed by the lenses of the eye-pieces are formed exactly as if the rays diverged from a real object, except that the rays from any point of the image do not fill the whole of the lens.

The centre of the object-glass is usually very distant as compared to the focal lengths of the lenses of the eye-piece. If we make x very large in the previous equation of condition, it becomes

$$\frac{\varpi}{f} + \frac{\varpi'}{f'} = \frac{a(\varpi + \varpi')}{ff'},$$

or

$$a = \frac{\varpi f' + \varpi' f}{\varpi + \varpi'}.$$

There is a special advantage in making the lenses of the same kind of glass, because then if we make two coloured images coincide, all the coloured images will be united. The condition for achromatism then becomes

$$a = \frac{f + f'}{2};$$

or in words, *the distance between the lenses must be half the sum of their focal lengths.*

191. The conditions for achromatism for any system of lenses, thick or thin, arranged along an axis, may easily be deduced from Gauss' theory.

For the relations between the coordinates of a point and its image may be written in the forms

$$k(\xi-a)(\xi'-a')+\mu'g(\xi-a)-\mu l(\xi'-a')-\mu\mu'h=0,$$

$$\frac{\eta}{\eta'}=\frac{\zeta}{\zeta'}=\frac{k(\xi'-a')+\mu'g}{\mu'},$$

using the notation of § 77.

If we suppose the point (ξ, η, ζ) to be fixed, for perfect achromatism the coordinates of the conjugate point ought to be independent of the particular ray chosen; and this for all values of ξ, η, ζ. These conditions may be fulfilled for two rays by making

$$\partial\left(\frac{\mu'g}{k}\right)=0, \qquad \partial\left(\frac{\mu l}{k}\right)=0, \qquad \partial\left(\frac{\mu\mu'h}{k}\right)=0,$$

$$\partial\left(\frac{k}{\mu'}\right)=0, \qquad \partial g=0.$$

These conditions are equivalent to

$$\partial g=0, \qquad \partial(\mu h)=0, \qquad \partial\left(\frac{k}{\mu'}\right)=0$$

and

$$\partial\left(\frac{\mu}{\mu'}l\right)=0.$$

The quantities g, h, k, l are connected by the equation

$$gh-hk=1$$

which may be written in the form

$$g\left(\frac{\mu l}{\mu'}\right)-\mu h\left(\frac{k}{\mu'}\right)=\frac{\mu}{\mu'};$$

from which we deduce the condition $\partial\left(\frac{\mu}{\mu'}\right)=0.$

This can only be realised perfectly by making the first and last media the same; this is actually the case in most optical instruments. The conditions then reduce to any three of the following

$$\partial g=0, \qquad \partial(\mu h)=0, \qquad \partial\left(\frac{k}{\mu}\right)=0, \qquad \partial l=0.$$

If we refer back to the values of the coordinates of the cardinal points of the system, it is easy to see that the preceding

conditions are equivalent to making the positions of the principal points, and the focal length of the system, the same for the two colours.

EXAMPLES.

1. Show that at a single refraction at a plane surface the dispersion is proportional to the tangent of the angle of refraction.

2. If two media be such that the increments of refracting indices for each species of homogeneous light be proportional to those indices themselves, a ray of light may be refracted at their common surface without dispersion.

3. If a ray be incident on one face of a triangular prism, and after entering the prism be reflected five times internally at the sides of the prism taken in order, show that it will emerge from the first face in a direction which makes the same angle with the normal as the incident ray, and that a pencil which so passes through the prism is not coloured.

4. If μ, ν be the indices of refraction for the red and violet rays, respectively, for crown-glass, and μ', ν' be the indices for the same rays for flint-glass; and if two thin lenses be constructed, one double convex of crown-glass with each surface of radius r, and one double concave of flint-glass with its surfaces of radii r and s, and they be placed in contact so that the light is incident on the surface of radius s; then the combination will be achromatic if $r+s : 2s = \mu - \nu : \mu' - \nu'$.

5. A small pencil of parallel rays of white light, after transmission in a principal plane through a prism, is received on a screen whose plane is perpendicular to the direction of the pencil; prove that the length of the spectrum will be proportional to

$$(\mu_v - \mu_r) \sin \iota \div \cos^2 D \cos (D + \iota - \phi) \cos \phi' ;$$

where ι is the refracting angle, ϕ, ϕ' the angles of incidence and refraction at the first surface, and D the deviation of the mean ray.

6. If an achromatic eye-piece for an astronomical telescope be composed of two convex lenses of different materials, prove that the distance between them must be intermediate between f' and $lf/(l-f)$, where f is the absolute focal length of the field-glass, f' that of the eye-glass, and l the length of the telescope from object-glass to field-glass.

7. Prove that a system of three thin convex lenses made of the same material, placed so that the distance between the first and second is a, and that between the second and third is b, is achromatic for an excentric pencil coming from a point on the axis whose distance from the first lens is

$$\frac{2abf_1 - af_1 f_3 - (a+b) f_1 f_2}{3ab - 2bf_1 - 2(a+b) f_2 - 2af_3 + f_2 f_3 + f_3 f_1 + f f_2}$$

when f_1, f_2, f_3 are the focal lengths (taken positively) of the three lenses, respectively.

8. In a spectroscope consisting of a single prism with collimator and telescope, the two last being fixed, and different parts of the spectrum being brought into the field of view by turning the prism, prove that for seeing two colours of refractive indices μ and $\mu + \delta\mu$, the prism must be turned through an angle

$$\frac{\Delta}{m-1},$$

where Δ is the dispersion of the colours, and m is the ratio of the breadth of the incident to the breadth of the emergent pencil of refractive index μ.

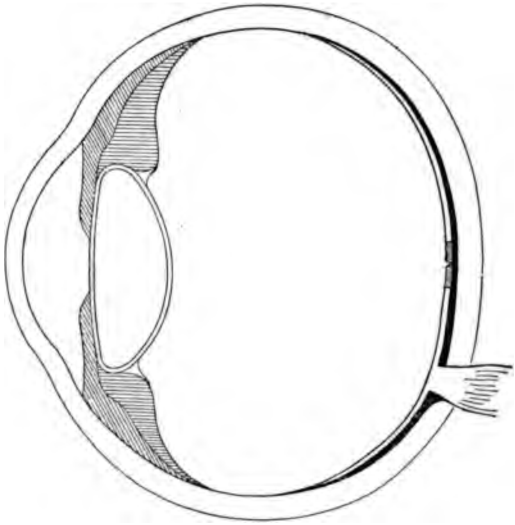
9. An opaque sphere, attached by a string to the bottom of a vessel containing clear transparent fluid, floats so that it is just immersed, the surface of the fluid being exposed to the rays of the sun, whose zenith distance is a. Describe the form of the shadow at the bottom, and the colours with which it is fringed.

Show that if the fluid be just sufficiently deep for the bottom to have no absolute shadow, its depth below the sphere : radius of sphere :: $\cos \frac{1}{2}(\theta + \theta')$: $\sin \frac{1}{2}(\theta - \theta')$, θ, θ' being the apparent zenith distances of the sun to an eye under water, for violet and red rays, respectively.

CHAPTER X.

THE EYE, AND VISION THROUGH LENSES.

192. THE eye is an optical instrument consisting essentially of a series of refracting media bounded by curved surfaces, and a delicate network of small nerve-fibres forming part of the optic nerve; a pencil of light incident upon the eye is refracted at the curved surfaces and brought to a focus on the network of nerve-fibres, and the impression is carried to the brain along the optic nerve.



The human eye is nearly spherical in shape, except in front, where it bulges out a little more than elsewhere. It is invested in a thick tough coat which, except in the small protuberant front part, is opaque and white and is called the *sclerotic*. This is

partly exposed in the living eye, and is in common language termed the white of the eye. The more protuberant part of the ball is covered with a thick, strong, transparent membrane called the *cornea*.

193. The eyeball has two other linings; immediately within the sclerotic is a thin membrane called the *choroïd*, and within that there is another thin lining called the *retina*.

The interior of the choroid coat is covered with black pigment, which gives it a velvety appearance; the function of this is to absorb rays of light which have passed through the retina and prevent them from being thrown back on the retina, so as to interfere with the distinctness of the images there formed. The anterior portion of the choroid, separating from the sclerotic, is thickened and forms the *iris*, which is a contractile curtain perforated in the centre by an aperture called the *pupil*. The outer edge of the iris is fixed, but the inner edge may be contracted by a strong muscular band running round it, and thus the size of the pupil may be changed. The use of the iris is to regulate the quantity of light allowed to fall on the sensitive part of the eye. In strong lights the pupil contracts automatically, and in feeble lights it is enlarged. The anterior surface of the iris is differently coloured in different persons, varying through all shades of blue, brown, and grey. The posterior surface is covered with black pigment, which serves to absorb any light which may fall upon it, due to internal reflexions or other causes.

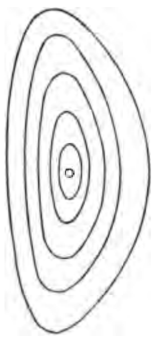
Just before separating from the sclerotic, the choroid splits into two layers; the anterior goes to form the iris, while the posterior is gathered into a circular plaited curtain which surrounds the outer edge of the lens (to be presently described) like a plaited collar. These plaits or folds, seventy to seventy-two in number, are called the *ciliary processes*. Beneath this dark plaited collar, and therefore in contact with the sclerotic, is a muscular collar, with radiating fibres, called the *ciliary muscle*.

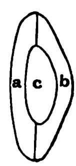
194. The retina is a delicate semi-transparent membrane resulting from the spreading out of the optic nerve, and is composed of the terminal fibres of this nerve and nerve cells; it covers the whole of the interior of the ball as far as the ciliary collar. Exactly in the centre of the retina is a round yellowish elevated spot,

about $\frac{1}{20}$th of an inch in diameter, having a minute indentation, called the *fovea centralis* at its summit. This is the point of distinct vision and the fovea centralis is the most sensitive part of the retina. About $\frac{1}{10}$th of an inch on the inner side of the yellow spot is the point at which the optic nerve spreads out its fibres to form the retina; this is the only spot on the retina which is not sensitive to light rays, and is known as the blind spot.

195. Within the eye, a little behind the iris, is suspended a soft transparent body, called the *crystalline lens*, of the form of a double convex lens, whose anterior surface is less curved than the posterior. The crystalline lens is contained in a thin transparent capsule, and is kept in its place by the ciliary processes. The substance of the lens is doubly-refracting, and acts like a uniaxal crystal cut perpendicular to its axis. It is composed of successive layers, whose refractive index increases towards the centre, its solid nucleus, which is of very small radius of curvature, refracting light most powerfully.

It is easy to see that the action of the lens is more powerful than if it were composed of homogeneous substance having the same





refractive index as the nucleus. For it may be regarded as the combination of a double convex lens c, with two other concave lenses a and b. These concave lenses will neutralize the effect of the lens c to a certain extent; but not so much as if their refractive indices were as high as that of c. The focal length of the lens may be found by experiment, and its shape being known, its so-called total refractive index may be found; that is, the refracting index which the lens would possess, were it homogeneous.

From what has been previously said, it follows that this total refractive index is greater than that of the nucleus.

The increase of refracting power from the outer portions to the inner portions of the lens serves partly to correct the aberration, by increasing the convergence of the central rays more than that of the extreme rays of the pencil.

196. The space between the cornea and the crystalline lens is filled with a transparent fluid resembling water, and thence termed the *aqueous humour.* The space between the crystalline lens and the retina is filled with another transparent fluid, somewhat more viscous than the former, and called the *vitreous humour.* These two humours, like the crystalline lens, are contained in transparent membranous capsules of great delicacy.

In their refractive indices the aqueous and vitreous humours differ very little from water, while the total refractive index of the crystalline lens is a little greater than that of water.

197. To determine the manner in which a pencil of light incident on the eye is refracted by it, we must know the refractive indices of the different media of which the eye is composed, and the forms and positions of the bounding surfaces. The refractive indices of the various media contained in the eye may be found in the usual way after they have been taken out from the eye. But it is found that the curvatures of the bounding surfaces undergo considerable changes after death, and the measurements have to be made from the eye of a living person. The details of these measurements cannot find a place here; but we may just indicate briefly the methods employed. The radius of curvature of a surface may be calculated if we know the magnitude of the image by reflection at it, of a body of given size and distance from it. The difficulty of measuring the magnitudes of these images arises from the continual movements of the eye. Helmholtz has invented an instrument, called an *ophthalmometer,* which may be used for this and many other similar purposes. This instrument depends on the fact that when an object is viewed through a thick glass plate so that the rays of light fall obliquely on the plate, the object appears to be of its natural size, but displaced a little to one side, the displacement being greater, the greater the angle of incidence

of the rays on the glass. The ophthalmometer consists of a telescope arranged for seeing at small distances, before whose object-glass is placed two thick plates of glass, so that one half of the object-glass looks through one plate and the other through the other. The plates can be turned about their common edge; when they are turned in opposite directions, two images appear, and the distance between these images may be calculated in terms of the inclinations of the plates to the axis of the telescopes. The plates are turned so that each image is displaced through half its length, and consequently the opposite terminals of the image coincide. The length of the image is then equal to the distance between the images.

Ex. If an object be viewed through a plate whose thickness is h, so that the rays are incident on the plate at an angle α; then if the corresponding angle of refraction be β, show that the apparent lateral displacement is

$$\frac{h \sin(\alpha - \beta)}{\cos\beta}. \qquad \text{(Helmholtz.)}$$

198. The distance between the surface of the cornea and the plane of the iris is measured by observing the image of a bright object as seen reflected by the cornea. The distance between the cornea and the bright image is known; by viewing this image from different positions, it is projected on the plane of the iris at different points, and by measuring the distances of these projections from the edge of the iris, the distance of the plane of the iris may be found. The other measurements are conducted in a similar manner. Further information on this subject may be found in Helmholtz' "*Handbuch der physiologischen Optik.*"

The anterior surface of the cornea is very nearly that of a segment of an ellipsoid of revolution, the axis of revolution being the major axis. The form of the posterior surface is not very accurately known. But the two surfaces of the cornea are very nearly parallel, and as the anterior surface is always moistened with water, whose refractive index is the same as that of the aqueous humour, the cornea acts like a plate of refracting medium, and produces no deviation in an incident ray. The cornea itself may therefore be entirely neglected, and we may for optical purposes suppose the aqueous humour extended to the anterior surface of the cornea.

The anterior surface of the crystalline lens is part of the surface of an oblate spheroid, and the posterior is supposed to be part of the surface of a paraboloid of revolution.

199. There are therefore three surfaces at which refraction takes place, the first surface of the cornea and the two surfaces of the crystalline lens. The centres of curvature of these surfaces are very nearly in a straight line, called the optic axis. For rays whose deviations from the axis are not large, the surfaces may be supposed to coincide with the spheres of curvature at their respective vertices. Gauss' theory of refraction at any number of spherical surfaces whose centres lie along an axis is therefore applicable to this case, and the positions of the focal points, the principal points, and the nodal points may be found by calculation, as soon as the radii of curvature, the positions of the refracting surfaces and the indices of refraction of the media are known. Listing has given the following numbers as representing very closely the constants of an average eye; in reckoning refractive indices, the refracting index of the air is taken to be unity.

(*a*) The radii of curvature of the bounding surfaces have the following values:

1. The anterior surface of the cornea.................. 8 mm.
2. The anterior surface of the lens...................10 mm.
3. The posterior surface of the lens.................. 6 mm.

(*b*) The distances between the refracting surfaces are:

From 1 to 2.. 4 mm.
From 2 to 3 (thickness of the lens)..................... 4 mm.
From 3 to the retina....................................13 mm.

(*c*) The indices of refraction are:

1. For the aqueous humour.............................. $\frac{103}{77}$,
2. For the lens (total)................................. $\frac{16}{11}$,
3. For the vitreous humour.............................. $\frac{103}{77}$.

From these data he calculates the positions of the cardinal points according to Gauss' theory, and finds that,

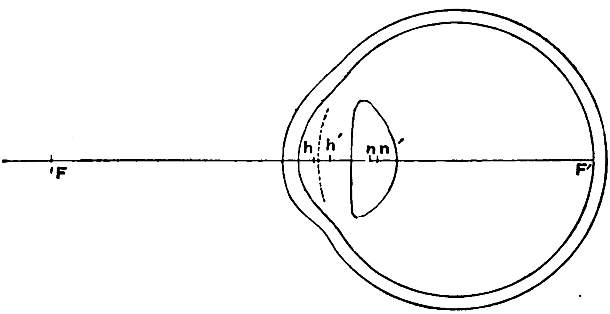
(1) The first focal point lies 12·8326 mm. in front of the cornea, and the second focal point 14·6470 mm. behind the posterior surface of the lens;

(2) The first principal point lies 2·1746 mm. and the second 2·5724 mm. behind the cornea, the distance between them being ·3978 mm.;

(3) The first nodal point lies ·7580 mm. and the second ·3602 mm. in front of the posterior surface of the lens.

(4) The first principal focal length is consequently 15·0072 mm. and the second 20·0746 mm.

The optical effect of this typical eye is found to agree very well with that of a natural eye; and considering the differences which



exist between individuals, the above numbers give as good an approximation to a natural eye as can be found.

200. It will be noticed that the two principal points lie very close together, as do also the two nodal points, so that without introducing much error, we may regard them as coinciding in each case. In this way we find a still simpler set of numbers, which correspond to what Listing calls the *reduced* eye. The single principal point lies 2·3448 mm. behind the cornea, and the nodal point ·4764 mm. in front of the second surface of the lens. Such an eye is exactly equivalent to a single refracting spherical surface, whose vertex is at the principal point and centre at the nodal point, the refractive index being $\frac{103}{77}$ as before. A point

and its image on the retina will lie on a line passing through the nodal point; and therefore if we wish to find in what direction lies a point whose image is in a given position on the retina, we have only to join the image to the nodal point and produce the line outwards.

201. When the eye is passive, it is clear that only the points which lie in a single surface will have images falling exactly on the retina. The form of this surface and its position may be determined from the optical constants of the eye. Any object lying on this surface will have an image on the retina similar to the original figure, but inverted, the lines joining corresponding points of the object and image all passing through the nodal point. But if a point does not lie on this surface, its image will be not on the retina, but in front of or behind it. In both cases the retina cuts the pencil of refracted rays not in a single point, but in a circle of diffused light. Hence it follows that an immoveable eye can only see distinctly objects lying in one surface, and if we consider only rays of light making small angles with the axis of the eye, this surface may be considered plane. All objects or portions of objects not lying in this plane give indistinct images, in which circles of diffusion correspond to luminous points of the object.

Experience teaches us, however, that an eye is capable of seeing distinctly at almost any distance; there must therefore exist an arrangement for altering the eye, and adapting it for seeing at different distances at will. The changes which occur as the result of this arrangement are included under the term *accommodation*. It is not known with absolute certainty for what distance an eye is adjusted when it is not actively accommodated, but it is almost universally supposed that a normal eye when passive is adjusted for objects at an infinite distance, so that the second focal point of the eye at rest is on the retina. It follows from this that accommodation only occurs in one direction, the eye being actively accommodated for near objects.

202. It has been found by experiment that accommodation is effected by change of form in the refracting surfaces of the eye. When the eye is accommodated for near objects, the anterior surface of the crystalline lens becomes more strongly curved, and approaches

nearer to the cornea; this is especially the case with the part which is not covered by the iris, and which arches forwards through the pupil. These changes are shown by the following experiment. If a lighted candle be placed at one side of the eye, in a room otherwise dark, and a person looks into the eye from the other side, three distinct small images of the flame are seen, due to reflexion at the refracting surfaces of the eye; the first, erect and virtual, is formed by the anterior surface of the cornea; the second, also erect, but much weaker and somewhat larger, is formed by the anterior surface of the lens; the third is brighter, inverted and real, and is formed by the second surface of the lens. If now the eye first look fixedly at a very distant object, and then at an object close to it, the second image becomes perceptibly smaller and approaches the first image, affording a proof that the anterior surface of the lens becomes more strongly curved and approaches the cornea.

Instead of a flame it is better to use a screen pierced with two holes, through which light is allowed to pass; the distance between the reflected images of these points may be measured by means of the ophthalmometer.

203. The following table exhibits the changes in the optical constants of the eye which occur during accommodation; the distances of the cardinal points are reckoned from the vertex of the cornea, and expressed in millimetres. The refractive indices of the media are the same as before.

	Eye at rest; adjusted for distant view.	Eye accommodated; adjusted for near view.
Radius of the curvature of the cornea	8	8
" " anterior surface of the lens	10	6
" " posterior surface of the lens	6	5·5
Position of the first surface of the lens	3·6	3·2
" second "	7·2	7·2
" first principal focus	−12·918	−11·241
" " principal point	1·9403	2·0330
" second "	2·3563	2·4919
" first nodal point	6·957	6·515
" second "	7·373	6·974
" second principal focus	22·231	20·248
First focal length	14·858	13·274
Second "	19·875	17·756

204. It has been seen that when the eye is at rest in any position and accommodated for an object, there is one point, the *fovea centralis*, where the vision is distinct, but that the vision is distinct only for a very small area about this spot. But the eye is usually in very rapid motion, and in an incredibly short space of time brings the various points of an object into distinct view. We are thus enabled to form a clear conception of a considerably extended object or surface. This is aided also by the duration of the impression produced by a light. It has been found by experiment that this duration depends on the character of the light. For strong lights Helmholtz gives $\frac{1}{24}$th of a second, and for weak lights $\frac{1}{10}$th of a second, as the duration of the impression. Lissajou and others assign about $\frac{1}{30}$th of a second as the lowest limit of the duration. If a spot on the retina be stimulated by a regular periodic light, whose period is sufficiently short, there will arise a continuous impression, which in intensity is equal to what would be produced were the whole incident light of any period uniformly distributed over the whole period.

205. If we consider an eye in a fixed position the field of distinct view is small, and the curvature of the retina over this field may be neglected, and the position of a point indicated by plane co-ordinates.

But the eye is capable of moving in its socket, so that its axis has a sweep of nearly 55° in every direction about its mean position; this solid angle within which any object can be seen by turning the eye, is called the *field of view*. The picture formed by external space to a single eye, might be perfectly represented by points and lines on a spherical surface, whose centre is the fixed point about which the eye turns; but it is more convenient in practice to project the objects seen upon a plane surface instead of a spherical surface. If we imagine ourselves standing at some distance from a large window reaching down to the ground, and if we suppose the objects seen by a single eye through the window to be painted upon the glass in the exact positions in which they are seen through the glass, such a representation will be a *perspective* view of the scene. The projection of any point, is the point where the plane of the window is met by a line joining the point to the centre of the eye. The projection of a straight line will be a

straight line; for we have only to draw a plane through the centre of the eye and the given straight line, and then this plane will meet the plane of the picture in a straight line, which will be the projection of the given line. If a series of lines meet in a point, their projections will meet in a point, this point being the projection of the given point. As a particular case, if the system of lines be parallel, their projections will meet in a point; to construct this point we must draw from the centre of the eye to the plane of the picture, a straight line parallel to the lines of the system; this point is called the *vanishing point* of the system. Vertical lines project into vertical lines. These remarks constitute the essential features in the theory of linear perspective; and from the principles here laid down it is easy to deduce the usual simple rules for constructing the projections of the outlines of given objects.

206. The retinæ of both our eyes receive impressions when we look at any external object and in certain positions of our eyes, we see two images, arising from the two retinæ, while in other positions we see only one image. To each point of one retina there is a *corresponding point* on the other; and when the images of an external point formed by the two eyes fall on corresponding points of the two retinæ, the point is seen single, but in other cases it is seen double. The points on the retina of an eye may be referred to two meridians formed on the retina by two planes through the axis of the eye. When the eye is directed forwards in a horizontal position, the points on the horizon have images lying on a meridian, which we may call the *retinal horizon*. Similarly certain lines appear vertical to an eye; the retinal image of these vertical lines is a meridian, which we may call the *apparently vertical meridian*. By experiment, Helmholtz concludes that the retinal horizon is actually horizontal for both eyes, but that the apparently vertical meridians are not quite perpendicular to the retinal horizon; they diverge outwards at their upper extremity. The inclination of each of these meridians to the real vertical is the same, and they include between them an angle varying from 2° 22′ to 2° 33′. Helmholtz also finds that in normal eyes, the points of distinct vision, as well as the retinal horizons and apparent verticals in the two eyes *correspond;* and further that points on the retinal horizons at

equal distances from the origins correspond, as do also points on the apparent verticals at equal distances from the origin. Now the field of distinct vision is not large, and we may regard the retinæ in the neighbourhood of the *fovea centralis* as nearly plane; then in each retina we are provided with a pair of natural oblique axes to which we may refer all other points. The eye is a very good judge of parallels, and therefore we may infer that on the retina parallels appear as parallels, so that from the previous remarks, it follows that points whose oblique co-ordinates in the two retinæ are the same, are corresponding points. Or, from the symmetrical arrangement of the axes, we may say that corresponding points are equally distant from each retinal horizon, and from each apparent vertical meridian.

207. Now whatever be the directions in which the two eyes are turned, there will always be points in the external world which are seen single; the locus of such points is called the *horopter*. We shall show that the horopter in general is a twisted cubic curve, formed by the intersection of two hyperboloids which have one generator in common.

In each eye let us take the optic axis for the axis of z, and retinal horizon as the axis of x, and use rectangular co-ordinates. Then if θ be the inclination of the apparent vertical to the horizon in one retina, $\pi - \theta$ will be that for the other. Let dashed letters refer to the second retina. Then the equations to the apparent verticals will be

$$x \sin\theta - y \cos\theta = 0,$$
$$x' \sin\theta + y' \cos\theta = 0.$$

Also the points (x, y) (x', y') will be corresponding points, provided that

$$y = y',$$
$$x \sin\theta - y \cos\theta = x' \sin\theta + y' \cos\theta.$$

Corresponding lines on the retinæ may then be expressed in the forms

$$l(x \sin\theta - y \cos\theta) + my + n = 0,$$
$$l(x' \sin\theta + y' \cos\theta) + my' + n = 0.$$

If corresponding points in the two retinæ be joined by lines

to the nodal points of the two eyes, those lines may be called *corresponding lines of vision.* If these lines produced outwards intersect each other, the point of intersection will be seen single. In the same way if corresponding retinal lines be joined by planes to the nodal points of the two eyes, these planes may be called *corresponding planes of vision.* Every pair of lines lying in corresponding planes of vision are seen single.

Let the co-ordinates of the nodal points be $x = 0$, $y = 0$, $z = c$, $x' = 0$, $y' = 0$, $z' = c$; then it is easily seen that the equations to a pair of corresponding planes of vision are of the forms,

$$l\,(x \sin\theta - y \cos\theta) + my + n\,(z - c) = 0,$$

$$l\,(x' \sin\theta + y' \cos\theta) + my' + n\,(z' - c) = 0.$$

For they pass through the nodal points, and meet the retinæ in corresponding lines.

So far, we have been taking a separate set of axes for each eye; we must however refer all points to one fixed system of axes in space. Suppose that on transformation, the expressions $x \sin\theta - y\cos\theta$, y, $z - c$, become P, Q, R where P, Q, R are linear functions of the co-ordinates, and that a similar notation be applied to the other co-ordinates; then the equations of a pair of corresponding planes of vision take the form

$$lP + mQ + nR = 0,$$

$$lP' + mQ' + nR' = 0.$$

From these equations it follows that through any point in space can be drawn one line which will be seen single. For in P, Q, R, P', Q', R' let the co-ordinates of the point be substituted, then the two equations serve to determine the ratios of $l : m : n$. These equations become indeterminate, however, if

$$PR' - P'R = 0,$$

$$QR' - Q'R = 0,$$

and by consequence, $$PQ' - P'Q = 0;$$

and where these conditions are satisfied, there can be drawn an infinite number of lines through the point which may be seen single, and therefore the point itself will be seen single. The last equations will therefore determine the horopter.

The equations

$$PR' - P'R = 0,$$
$$QR' - Q'R = 0,$$

represent hyperboloids; and they have a generator in common, determined by the equations

$$\left.\begin{array}{l} R = 0 \\ R' = 0 \end{array}\right\}.$$

If we eliminate R' between the equations to the hyperboloids, we get

$$(PQ' - P'Q)\, R = 0,$$

and therefore if R be not zero, we get the third equation

$$PQ' - P'Q = 0.$$

The points on the twisted cubic determined by the first two equations will therefore also satisfy the third equation. The horopter will therefore consist of a twisted cubic. The forms of the equations show that the curve passes through the nodal points of both the eyes.

The line joining any two points of the horopter will be seen single; it follows therefore that through any point of the horopter an infinite number of lines may be drawn which will be seen single, and that these lines lie on a cone of the second order whose vertex is the point.

208. There is one arrangement of the eyes in which the horopter is a plane surface; this is when both eyes are directed horizontally towards the horizon. In this case we may take as our origin the point midway between the eyes in the plane of the retinæ, the axis of z in the given horizontal direction parallel to the axes of both eyes, and the axis of y vertical. Let $2a$ be the distance between the eyes, and c the distance of the nodal points from the retinæ. Then the equation of a pair of corresponding planes of vision will be

$$l\,\{(x - a)\sin\theta - y\cos\theta\} + my + n\,(z - c) = 0,$$
$$l\,\{(x + a)\sin\theta + y\cos\theta\} + my + n\,(z - c) = 0.$$

These are satisfied together for all values of $l : m : n$, provided that

$$y\cos\theta + a\sin\theta = 0.$$

The equation to the horopter is therefore

$$y = -a \tan \theta.$$

This is *a plane parallel to the horizon and passing through the point where the apparently vertical meridians will meet if produced.*

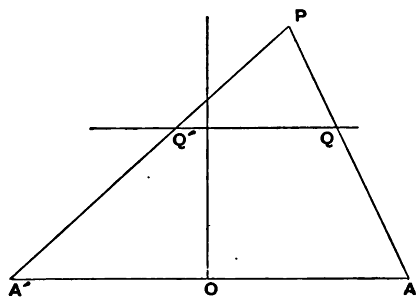
Helmholtz finds that for an average man this plane very nearly coincides with the ground on which he stands, and it seems not improbable that the cause of the obliquity of the apparent vertical meridians may depend upon this relation of the ground to the eyes.

209. Our most accurate estimate of the distances of visible objects depends upon our having two eyes. As we fix our gaze successively upon points at different distances we have to change the convergence of the axes of the two eyes, and from the degree of convergence of these axes when we look at any point we form an estimate of the distance of the point. Distances can however be estimated by a single eye, by observing the relative changes of position of objects, when the observer's position is changed.

Our idea of solidity also depends upon vision with two eyes. The views presented to the two eyes are slightly different, because the eyes have slightly different positions; and it is by the blending of the two impressions received upon the two retinæ that we receive the idea of solidity. This can be well shown by aid of the stereoscope. This instrument was invented by Wheatstone for the purpose of combining two different photographic pictures, one of which is presented to each eye. These pictures are not exactly alike, but are taken by a camera with two lenses placed a small distance apart, so that they represent two different views such as might be presented to two eyes observing the scene. By means of mirrors or prisms the pictures are seen superimposed, and the impression produced on the mind by these superimposed views is exactly the same as if we were looking at the real scene, each object appearing in relief as it would in nature. For a perfect stereoscopic representation, the points at an infinite distance must fall on corresponding points of the two retinæ when the axes of the eyes are parallel. If the pictures are brought nearer to each other in the same plane than in the positions thus determined, the impression produced is exactly that of a relief picture.

210. To find the changes produced by making a pair of stereoscopic pictures approach each other, let us take axes and use the methods of analytical geometry.

Let A, A' be the centres of the two eyes, distant $2a$ from each other.



Take O, the middle point of AA' as origin, and OA as the positive direction of the axis of x, and a horizontal line through O as the axis of z. Let P be any point, and Q, Q' the points where the lines AP, $A'P$ meet a plane $z=b$; then Q, Q' are the stereoscopic projections of the point P. Let (ξ, η, ζ) be the coordinates of P, (u, v, b) (u', v', b) the coordinates of Q and Q', respectively. Then the equation of AP is

$$\frac{x-a}{\xi-a}=\frac{y}{\eta}=\frac{z}{\zeta},$$

and therefore, if we put $z=b$, we get the values of u, v, namely,

$$u=a+\frac{b}{\zeta}(\xi-a),$$

$$v=\frac{b}{\zeta}\eta.$$

Similarly, if we change the sign of a throughout, we find,

$$u'=-a+\frac{b}{\zeta}(\xi+a),$$

$$v'=\frac{b}{\zeta}\eta.$$

Combining these equations, we get

$$2a+u'-u=\frac{2ab}{\zeta}=e, \text{ say.}$$

This quantity e is called the *stereoscopic difference*, and is constant for points lying in any plane perpendicular to Oz, and vanishes for points at infinity.

When we know the coordinates of the projections Q, Q', those of the point P are known. For from the preceding equations, if we denote by x the arithmetic mean of u and u', we find the equations,

$$\left.\begin{aligned} \xi &= x \,.\, \frac{2a}{e} \\ \eta &= v \,.\, \frac{2a}{e} \\ \zeta &= b \,.\, \frac{2a}{e} \end{aligned}\right\},$$

or

$$\left.\begin{aligned} \frac{\xi}{\zeta} &= \frac{x}{b} \\ \frac{\eta}{\zeta} &= \frac{v}{b} \\ \frac{1}{\zeta} &= \frac{e}{2ba} \end{aligned}\right\}.$$

211. Now suppose that the right-hand picture is moved towards the left and the left-hand picture towards the right, each through a distance η. Then x will be unchanged, and e will become $e + 2\eta$. Also we will suppose the scale of the original drawing diminished in the proportion of $1 : n$, so that we must write $n\xi$, $n\eta$, $n\zeta$ for ξ, η, ζ, respectively. Then if we denote the point corresponding to (ξ, η, ζ) after the transformation, by the coordinates (ξ', η', ζ'),

$$\frac{\xi'}{\zeta'} = \frac{x}{b} = \frac{\xi}{\zeta},$$

$$\frac{\eta'}{\zeta'} = \frac{v}{b} = \frac{\eta}{\zeta},$$

$$\frac{1}{\zeta'} = \frac{e + 2\eta}{2ab} = \frac{1}{n\zeta} + \frac{\eta}{ab}.$$

For brevity let $\dfrac{ab}{\eta} = p$; then the last equation becomes

$$\frac{1}{\zeta'} = \frac{1}{n\zeta} + \frac{1}{p}.$$

From this it follows that all points originally at infinity correspond after transformation to points on the plane

$$z = p.$$

This plane has been called the *principal plane* of the relief-picture.

Any plane in the original system will transform into a plane. For the equation

$$A\xi + B\eta + C\zeta + D = 0,$$

when expressed in terms of (ξ', η', ζ'), becomes

$$A\frac{\xi'}{\zeta'} + B\frac{\eta'}{\zeta'} + C + D\left\{\frac{n}{\zeta'} - \frac{n}{p}\right\} = 0,$$

or

$$A\xi' + B\eta' + \left(C - \frac{nD}{p}\right)\zeta' + Dn = 0.$$

It may easily be shown that a series of planes passing through a line transforms into another series of planes passing through a line.

A series of parallel planes is obtained by making D a variable parameter. The transformed planes will then clearly pass through the line represented by the equations

$$\left.\begin{aligned} A\xi' + B\eta' + C\zeta' &= 0 \\ \zeta' &= p \end{aligned}\right\}.$$

Thus a series of parallel planes is transformed into a series of planes passing through a line which lies in the principal plane.

Also every plane through the origin remains unaltered by transformation; this is clear if we make $D = 0$, in the previous equations.

From these theorems it easily follows that a series of lines meeting in a point will be transformed into another series of lines meeting in a point, and a series of parallel lines into a system of lines meeting in a point on the principal plane.

Further, from the equation

$$\frac{1}{\zeta'} = \frac{1}{n\zeta} + \frac{1}{p}$$

we get $\zeta = \zeta'$, provided that

$$\frac{1}{\zeta}\left\{\frac{n-1}{n}\right\} = \frac{1}{p},$$

or
$$\zeta = \frac{p(n-1)}{n}.$$

Also, in the same case, $\xi = \xi'$, $\eta = \eta'$; so that every point on the plane

$$\zeta = p\left\{1 - \frac{1}{n}\right\}$$

transforms into itself. This plane is called the *congruence plane.*

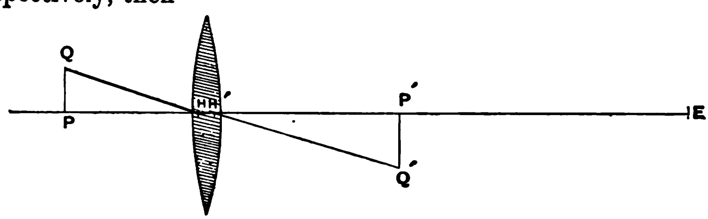
When the congruence plane approaches indefinitely near to the principal plane, that is, when n becomes indefinitely large, the relief picture becomes a plane perspective drawing.

A relief picture, with much smaller distances and smaller depth than the original, will produce the impression of the original in form and dimensions, even to binocular vision, and therefore will be a much more perfect imitation, at least as regards the form of the object, than any plane picture could be.

Vision through a lens.

212. *To find the visual angle under which an object is seen through a lens, by an eye situated on the axis of the lens.*

Let β, β' denote the linear magnitudes of the image and object, x, x' their distances from the first and second principal points, respectively, then



$$\beta' = -\beta \frac{x'}{x}.$$

Let ξ denote the distance of the eye from the second principal point, so that the distance of the image from the eye is $\xi - x'$; then if θ be the angle under which the part of the image lying on one side of the axis is seen by the eye

$$\tan\theta = \frac{\beta'}{\xi - x'} = \frac{\beta}{x\left(1 - \frac{\xi}{x'}\right)}.$$

The angle θ may therefore be increased so as to be brought as nearly equal to a right angle as we please, by making ξ nearly equal to x'. The nature of vision, however, imposes a limit, because the eye is not capable of distinct vision when the point lies within a certain distance. If λ denote the least distance for distinct vision, then the greatest value of θ is found by putting $\xi - x' = \lambda$; we therefore get

$$\tan\theta = -\frac{\beta}{\lambda}\frac{x'}{x}.$$

The negative sign indicates that the image will be inverted.

213. When the image falls on the other side of the lens, x' will be negative; and then $\tan\theta$ will be made as large as possible by bringing the eye close to the lens. In this case ξ is so small that it may usually be neglected, and we get $\tan\theta = \beta/x$, nearly.

From this it appears that $\tan\theta$ may be made as large as we please by diminishing x; but there is a limit to the possible diminution of x, for x' must not be sensibly less than λ. Now

$$\frac{1}{x} + \frac{1}{x'} = \frac{1}{f},$$

and therefore putting $x' = -\lambda$, we get

$$\frac{1}{x} = \frac{1}{\lambda} + \frac{1}{f}.$$

The greatest visual angle under which an object may be seen distinctly, will therefore be given by the formula

$$\tan\theta = \beta\left(\frac{1}{\lambda} + \frac{1}{f}\right).$$

The tangent of the angle subtended at the eye by the object when placed in the position of the image is β/λ; the ratio of these tangents is the *magnifying power;* and therefore, denoting the magnifying power by m,

$$m = 1 + \frac{\lambda}{f}.$$

In convex lenses f is positive, and therefore the object will appear magnified; in concave lenses f is negative, and therefore the object appears to be diminished by the lens.

214. If in the formula for tan θ we substitute for x' its value in terms of x, it becomes

$$\tan \theta = \frac{\beta}{x + \xi - \frac{x\xi}{f}}.$$

By giving different values of x, ξ, it is easy to see from this formula, in what manner the visual angle changes when the positions of the object and eye vary.

We notice that this formula is symmetrical with regard to x and ξ, so that the positions of the eye and object may be interchanged in every case without altering the visual angle.

When the eye is at a principal focus the apparent magnitude is independent of the position of the object; and similarly when the object is at a principal focus its apparent magnitude is independent of the position of the eye; the apparent magnitude being, in both cases, equal to that under which the object would be seen by the naked eye, when at a distance equal to the focal length of the lens. For if we make either $x = f$, or $\xi = f$, we get $\tan \theta = \beta/f$.

Again, when either the eye or the object be close to the lens, the apparent magnitude is that under which the object would be seen by the naked eye. For in these cases, we must make ξ or x very small; and therefore $\tan \theta = \beta/x$ or β/ξ.

215. But in all cases the visual angle will necessarily be limited by the aperture of the lens; so that the greatest value of $\tan \theta$ will be equal to y/ξ nearly, where y denotes the semi-aperture of the lens. The greatest linear extent of object, visible through a lens in any position, may be called the *field of view*. Its magnitude is at once ascertained by equating the value of $\tan \theta$ as previously found to y/ξ; the greatest value of β is accordingly

$$\beta = yx\left(\frac{1}{x} + \frac{1}{\xi} - \frac{1}{f}\right).$$

The linear extent of the field of view, therefore, varies as the aperture of the lens, other things remaining the same.

When the object is in the principal focus of the lens $x = f$, and therefore $\beta = yf/\xi$. When the eye is in the principal focus of the lens, the linear extent is equal to the aperture of the lens, whatever the position of the object. For if $\xi = f$, $\beta = y$.

When the object is close to the lens, that is, when x is very small, the value of β becomes very nearly equal to y; so that the extent of the field is in this case independent of the position of the eye. On the other hand, when the eye is close to the lens, that is, when ξ is small, the field becomes very great.

Spectacles and Reading Glasses.

216. The distinctness of objects as seen by the naked eye depends on the accurate convergence of the rays of different pencils to points on the retina. We have seen that the eye is furnished with a mechanism for adapting itself for seeing distinctly objects at different distances. A normal eye when not actively accommodated is adapted for rays coming from a distant object, or for parallel rays; and it must be accommodated for seeing objects which are near, the range of distinct vision extending from five or six inches to infinity. Eyes for which the greatest distance of distinct vision is finite are called *short-sighted*, or *myopic*; these eyes can only bring divergent pencils to a focus on the retina. On the other hand, eyes which can bring to a focus on the retina not only parallel rays but convergent pencils are called *long-sighted* or *hypermetropic*. The defects in these eyes depend on the length of the axes of the eyes; in a short-sighted eye, the axis is too long, and in a long-sighted eye it is too short. In both short-sighted and long-sighted eyes the accommodating mechanism may be quite perfect. When this is the case, their defects may be entirely remedied and the eyes made normal by the use of spectacles.

217. Let the range of distinct vision by the naked eye extend from points distant a, b from the eye. In a normal eye, b will be infinite; in a short-sighted eye b will be finite and positive, and in a long-sighted eye b will be finite and negative. Suppose the eye to view an object through a lens of focal length f, placed close to the eye, f being positive for a collective lens, and negative for a dispersive lens. Then if x, x' be the distances from the eye (or from the lens) of an object and its image, respectively, measured outwards,

$$\frac{1}{x} - \frac{1}{x'} = \frac{1}{f}.$$

The rays striking the eye will appear to diverge from the image; and therefore the rays may be brought to a focus provided x' lies between the limits a, b. If we substitute for x' the values a, b in succession, the corresponding values of x will be the limits of the range of distinct vision through the spectacles.

When the accommodating mechanism is perfect, we have only to choose f, so that the farther limit of distinct vision is at an infinite distance. We must therefore make x infinite when $x' = b$, and thus we find the focal length of the spectacle glass, namely, $f = -b$. The nearer limit of the range of distinct vision becomes

$$x = \frac{ab}{b-a};$$

and therefore the range of distinct vision through the spectacles will extend from $ab/(b-a)$ to infinity.

In a short-sighted person b is finite and positive, and therefore f is negative; he must therefore use dispersive lenses, generally double concave lenses, whose focal length is equal to the greatest distance of distinct vision by the naked eye. Thus if the range of distinct vision extends from 3 to 6 inches from the eye, the use of a concave lens whose focal length is 6 inches, will cause the range of distinct vision to extend from 6 inches to infinity.

On the other hand, in a long-sighted eye b is negative and therefore f is positive. For example, if the range of distinct vision extend from 12 inches outwards through infinity to -12 inches, the spectacles chosen must be collective lenses of 12 inches focal length; substituting these values in the general formula, we find that the range is then from 6 inches to infinity.

Practically, these glasses may be chosen by making the person look at a distant object; then the weakest concave glasses which will enable a short-sighted person to see this object distinctly, and the strongest convex glasses which will enable a long-sighted person to see it distinctly, are the glasses suitable to the eyes.

The limiting points of the range of distinct vision may be measured by making the person look through suitably chosen convex lenses, so that the points in question are brought within 12 inches from the eye, and then their distances can be measured on a divided scale. They are generally not the same for both eyes, so that the two eyes require different glasses.

Short-sighted persons who have to do delicate work, have sometimes to bring things close to the eyes; in this case they should use rather weaker concave glasses, than those prescribed above. For the same purpose, achromatised prismatic glasses, which are thicker towards the sides next the nose, and thinner towards the sides next the temples are used, because the objects can then be seen with less convergence of the axes of the eyes.

218. As the age of a person advances, the eye gradually loses its power of accommodation; it is supposed that the outer layers of the crystalline lens lose their elasticity, so that the lens becomes less capable of changing its form and curvature. This defect is known as *presbyopia*. It is entirely different from the defect described above, called long-sightedness; though aged persons are sometimes said to be long-sighted. The structure of an eye does not alter with age, so that a person with normal eyes, will still see distant objects when he becomes old; but the range of accommodation of the eye is then less than before, so that it cannot bring to a focus on the retina pencils of rays issuing from points very near to it; in other words, the nearer limit of distinct vision has receded from the eye. Presbyopic eyes therefore need convex glasses to enable them to see near objects, as in reading or writing; but they must be laid aside to look across a room or at a distant view. Usually the glasses are chosen so as to bring the nearer limit of distinct vision to 10 or 12 inches from the eye. For very aged persons, whose sight has lost its keenness, it is sometimes advisable to use spectacles which will bring this nearer limit to within 8 or even 7 inches from the eye, so that objects may be seen under a greater angle.

From what has been said, it is evident that presbyopia may exist along with the other defects previously mentioned. Both long-sighted eyes and short-sighted eyes can be made normal by the use of spectacles, as we have seen. When presbyopia sets in, these eyes will need two pairs of spectacles, one for walking, and another for reading and writing.

219. A convex lens of considerable aperture and magnifying power is often used as a reading glass, or for viewing the details of small objects. Such a glass may be used by both short and long-sighted people. For suppose that the glass is placed, so that the object is in the principal focus of the glass, then the rays

emerging from the lens are parallel. If the glass be now moved a little nearer to the object, the emergent rays will diverge, and can be brought to a focus on the retina by a short-sighted eye; if on the other hand the glass be moved a little farther away from the object, the emergent rays will converge and will be adapted for distinct vision by a long-sighted eye. In each case the magnifying power will be $1 + \lambda/f$, where λ is the least distance of distinct vision, and f the focal length of the lens.

Astigmatism.

220. In addition to the defects already mentioned, the eye is often not symmetrical about its axis, and also, like other optical instruments it is subject to spherical aberration. The defects of the image arising from these causes are all included in the term *astigmatism.*

If a person look at a distant small bright point, while his eyes are accommodated for a nearer object, we should expect that there would be a small circular patch of light on the retina; but often, instead of a circular spot of light, he sees a star-shaped figure, with from four to eight rays. The shape of this figure is different in different individuals, and generally different for the two eyes. If the light be white light, the parts near the edges of the bright parts of the figure are tinged with blue, while those in the centre are a reddish yellow. If the light be weak, only the bright parts of the figure are visible and several images of the bright point are seen. The appearance of a star or a distant light, as a figure with streaming rays, is connected with these appearances. Similarly if the eye be accommodated for an object at a greater distance than the bright point, another star-shaped figure is seen; and even if the eye can accommodate itself to the light, provided the light be sufficiently strong, the same appearances present themselves. If the object be a fine line of light, the appearances can easily be deduced from the preceding phenomena; several images of the line are seen, most eyes seeing at least two.

Some of these phenomena are due to moisture on the cornea, and can be changed by blinking with the eye-lids. But most of them are caused by irregularities in the crystalline lens. Donders

investigated them by allowing light to pass into the eye through a small slit, which he moved into different meridian planes. He found that usually every sector of the lens brought the incident rays to a focus, but that the foci for different sectors did not coincide. Also each sector did not bring the rays *very accurately* to a focus, but those rays which were incident near the axis of the eye, had their foci farther off than those which were incident near the edges of the pupil.

These phenomena are called *irregular astigmatism*, and they cannot be removed.

221. The defects of the image, arising from want of symmetry in the curvature of the refracting surfaces, and especially of the cornea, are called *regular astigmatism*. Regular astigmatism occurs in almost all eyes. The eye is not in general accurately accommodated for horizontal and vertical lines at the same time. If a person look at a figure consisting of a series of lines all radiating from a point, when any one of the lines looks clear and sharply defined, the others do not, and as the distance of the figure changes, first one line and then another is well defined. Or if he look at a figure consisting of a series of concentric circles drawn at small intervals, a peculiar star-shaped appearance is present in the figure. In the white rays, the edges between the black lines and the white intervals are clear and well defined, and they are less and less clear till we reach the blacker rays. If the accommodation of the eye be allowed to change or the figure be moved to a greater or less distance from the eye, other parts of the figure become clear, and it seems as if the clear rays moved to and fro. This may be explained by the theory of thin pencils already developed. If we trace back a thin pencil from a point on the retina through the eye into the air, the emergent pencil will be orthogonal, and will have two focal lines at right angles to the axis of the pencil, lying in planes through the axis of the pencil which are perpendicular to each other. For the state of accommodation of the eye, the positions of these focal lines are the best positions for seeing lines parallel to their directions respectively. If the distances of these focal lines from the eye be p, q, then $1/p - 1/q$ may be taken as a measure of the astigmatism of the eye in that state. If a lens be placed in front of the eye, it will not change the value of $1/p - 1/q$; so that

we may assume that $1/p - 1/q$ retains very nearly the same value when the state of accommodation in the eye changes.

222. This astigmatism may be corrected by a properly chosen astigmatic lens. If we place before the eye a lens such that when a pencil of light issuing from a bright point pass through it, the emergent pencil has a pair of focal lines coinciding with those already found for the eye, then when the pencil enters the eye it will be refracted so that the rays meet in a point.

The lenses used for this purpose have one surface cylindrical. It was shown in § 157 that the measure of the astigmatism of a cylindrical lens, the radii of whose surfaces are a, and b, and the angle between the generators of the two cylinders 2α, is

$$\frac{1}{u} - \frac{1}{v} = (\mu - 1)\sqrt{\frac{1}{a^2} + \frac{1}{b^2} - \frac{2}{ab}\cos 4\alpha},$$

or, in the case when the posterior surface is plane,

$$\frac{1}{u} - \frac{1}{v} = \frac{(\mu - 1)}{a}.$$

A lens of this latter form may always be chosen to cure the astigmatism of the eye; the radius of the cylinder must be such that

$$\frac{1}{p} - \frac{1}{q} = \frac{\mu - 1}{a}.$$

223. If two astigmatic lenses, one concave and the other convex, be united so that their plane surfaces coincide, we reproduce the conditions of the first case. When the radii of the cylindrical surfaces be equal, $a = b$, and therefore

$$\frac{1}{u} - \frac{1}{v} = \frac{2(\mu - 1)\sin 2\alpha}{a}.$$

The lenses may be arranged so that they can be turned about their common axis, so that α may be made to vary; then the astigmatism of the combination may be made to vary from zero, when $2\alpha = 0$, up to $2(\mu - 1)/a$, when $2\alpha = \frac{1}{2}\pi$.

This furnishes a very simple method of measuring the astigmatism of an eye. The person is made to view through a convex lens a figure formed by a series of lines diverging from a point, and the figure is moved farther and farther away till the farthest point is reached for which the spectator can see any of the

lines distinctly. A pair of crossed astigmatic lenses is then interposed, and turned about their common axis until all the lines are visible; the astigmatism of the eye is then corrected by that of the combination; and the measure of astigmatism is determined by observing the angle between the generating lines of the cylindrical surfaces.

224. The direction of the focal lines of the eye are determined by observation; if we choose the axes of co-ordinates to coincide with the focal planes, $\theta = 0$, using again the notation of § 157. Hence $\sin 2\alpha = 0$, which gives $\alpha = 0$ or $\frac{1}{2}\pi$.

If we choose the generating line of the cylindrical surface to coincide in direction with the more distant focal line, $\alpha = \frac{1}{2}\pi$, and

$$\frac{1}{p} - \frac{1}{q} = -\frac{(\mu - 1)}{a}.$$

This shows that a is negative, and therefore that the cylindrical surface is convex.

If this surface is to be concave, we must make $\alpha = 0$, or the direction of the generators of the cylindrical surface must coincide with that of the nearer focal line.

The second surface of the spectacle glass, instead of being plane may be made spherical. This is equivalent to uniting the astigmatic lens with an ordinary lens one surface of which is plane. This will not alter the measure of the astigmatism, and by choosing the curvature of the spherical surface properly, the lens may be made to correct astigmatism and also myopia or presbyopia at the same time.

The use of cylindrical lenses to cure the defects of astigmatism was discovered by Airy.

On vision through any number of lenses.

225. To determine in what manner an object will be seen through any system of lenses, it will be necessary to trace the course of a pencil of rays from any one point of the object; the several foci of this pencil determine the positions of the several images of the object. The field of view, the effective apertures of the lenses, the visual angle under which the object is seen, and the best position for placing the eye, are all found by considering the course of the *axes* of these several pencils. These axes all

pass through the nodal points of the object-glass, or the first lens of the system, and may therefore be considered as a pencil of rays issuing from the second nodal point of this lens; the extreme rays of this pencil determine the field of view and the effective apertures of the several lenses after the first.

We have already investigated the position of the focus of a small pencil issuing from a point and traversing any number of lenses bounded by spherical surfaces. In such an optical system the initial and final media are the same, and therefore the principal points coincide with the nodal points, and the two focal lengths are equal.

226. The magnifying power of any telescope may be defined to be the ratio of the angle under which an object is seen through the telescope, to the angle under which it would be seen by the naked eye. This definition supposes that the *linear* dimensions of the object and image are to be compared, and not their areas. This is sometimes expressed by saying that an instrument magnifies a certain number of *diameters*.

The axes of the extreme pencils which enter the object-glass, determine the angle under which the object would be seen, were the eye placed at the centre of the object-glass; and in the case of a telescope, this does not differ sensibly from the angle under which it would be seen by the eye in its position when looking through the instrument; for the distance of an object is usually very great compared with the length of the telescope.

With the notation which we used in stating Gauss' general theory, let the axis of one of the extreme pencils be determined by the quantities β, b; and after refraction at the several surfaces, by $\beta_1, b_1, \beta_2, b_2 \ldots \beta', b'$. Then since the axes of all the incident pencils pass through the first nodal point of the object-glass, $b = 0$, very nearly; for the first nodal point will be very close to the surface of the object-glass. Substituting this value of b in the equation

$$\beta' = kb + l\beta,$$

we find that

$$\frac{\beta'}{\beta} = l,$$

and therefore *the magnifying power of the instrument is represented by l.*

227. The image of the surface of the object-glass as seen through the instrument is called the *eye-ring*. Every ray which passes through the instrument will emerge within the eye-ring, at the image, namely, of the point at which the ray strikes the object-glass.

If the instrument be directed to an illuminated surface, or to the sky, each point of the eye-ring receives light from all points of space whose rays can traverse the instrument, that is, from all points of space which can be seen by help of the instrument. If the eye be placed so that its centre is at, or close to, the centre of the eye-ring, it will therefore embrace the entire field of the instrument. The centre of the eye-ring, is therefore the best position for the eye, and is called the *eye-point*.

We have seen that the axes of the several pencils striking the object-glass form a pencil issuing from the second nodal point of the object-glass; the final focus of this pencil as it passes through the instrument, will be very near to the centre of the eye-ring; and therefore the eye will receive as many of the axes of pencils issuing from the object as possible.

To find the magnitude and position of the eye-ring, we have only to make $\xi = a$; then

$$\xi' = a' - \frac{\mu' h}{l}.$$

Also for this point,

$$\frac{\mu'\eta'}{\mu\eta} = \frac{\mu' g - \mu' \dfrac{kh}{l}}{\mu},$$

so that $$\frac{\eta'}{\eta} = \frac{1}{l}, \text{ since } gl - kh = 1.$$

Remembering that l is the magnifying power of the telescope, this equation shows that *the magnifying power of the instrument is equal to the ratio of the radius of the object-glass, to that of its image as seen through the telescope.*

This gives a practical way of measuring the magnifying power of a telescope. The telescope is pointed to a bright surface, and the diameter of the eye-ring is measured by a graduated scale and lens, forming a micrometer. The diameter of the object-glass can also be measured, and the ratio of the latter to the former gives the magnifying power.

228. The *Field of view* is determined by the axes of the extreme pencils which meet the object-glass. The effective apertures of the various refracting surfaces may be obtained by finding the corresponding values of b. If β be the inclination of the extreme axes of the incident pencils, then, since b vanishes, the effective apertures will be determined by the equations,

$$\begin{array}{ll} \beta_1 = \beta, & b_1 = \beta_1 t_1, \\ \beta_2 = \beta_1 + k_1 b_1, & b_2 = b_1 + \beta_2 t_2, \\ \beta_3 = \beta_2 + k_2 b_2, & b_3 = b_2 + \beta_3 t_3, \\ \dots\dots\dots\dots & \dots\dots\dots\dots \\ \beta' = \beta_{n-1} + k_{n-1} b_{n-1} & b' = b_{n-2} + \beta_{n-1} t_{n-1}. \end{array}$$

By these equations $b_1, b_2 \dots b'$ are determined.

If we add the first set of equations, we arrive at the result,

$$\beta' - \beta = k_1 b_1 + k_2 b_2 + \dots + k_{n-1} b_{n-1},$$

which may be written,

$$(l-1)\beta = k_1 b_1 + k_2 b_2 + \dots + k_{n-1} b_{n-1}.$$

This shows that *the field of view is continually increased by adding more convex lenses;* for corresponding to both surfaces of a convex lens k is positive.

If any lens have its aperture diminished, the values of all the other apertures and of the field of view are diminished in the same proportion. It is useless to make the aperture of any of the lenses greater than its effective aperture.

229. Sometimes it happens that the position of the eye-point falls within the telescope, that is, in front of the outer surface of the eye-lens; this will be the case when h/l is positive, for then $\xi' - a'$ will be negative. The eye cannot then be placed at the eye-point, but is put as close to the eye-point as possible; it will therefore be placed close to the eye-lens. In estimating the field of view, the radius of the pupil must then be used instead of the radius of the outer surface of the eye-lens.

Brightness of Images.

230. It has already been proved that if β, β' be the linear dimensions of an object and its image by refraction at a spherical

surface separating two media of refractive indices μ, μ', and if α, α' be the angles of divergence from the axis of two corresponding rays in the two media, then

$$\mu\beta \tan\alpha = \mu'\beta' \tan\alpha'.$$

In other words, the value of $\mu\beta \tan\alpha$ is unchanged by the refraction; and therefore after any number of such refractions the same law must hold; that is,

$$\mu\beta \tan\alpha = \mu'\beta' \tan\alpha',$$

where μ', β', α', refer to the final medium.

To the first order of small quantities, we may write α for $\tan\alpha$, α' for $\tan\alpha'$, and therefore the formula becomes,

$$\mu\alpha\beta = \mu'\alpha'\beta'.$$

Now let dS, dS', be small elements of the object and image, standing perpendicular to the axis, so that

$$dS : dS' = \beta^2 : \beta'^2;$$

and let $d\omega$ be the small solid angle bounding a pencil diverging from dS, and let $d\omega'$ be the corresponding solid angle for the emergent pencil. Then if I be the brightness of the object, the quantity of light issuing from the object is,

$$L = IdS\,d\omega;$$

and if I' be the brightness of the image, the quantity of light emerging from the system will be

$$L' = I'dS'\,\delta\omega'.$$

On the supposition of no absorption of light by the media, $L = L'$, and therefore

$$IdS\,d\omega = I'dS'\,d\omega'.$$

But $\qquad dS : dS' = \beta^2 : \beta'^2,$

and also $\qquad d\omega : d\omega' = \alpha^2 : \alpha'^2.$

Hence $\qquad I\alpha^2\beta^2 = I'\alpha'^2\beta'^2.$

Referring back to the preceding equation, connecting α, β with α', β' we arrive at the equation

$$I : \mu^2 = I' : \mu'^2.$$

If the initial and final media are the same, as is nearly always the case with optical instruments,

$$I = I';$$

that is, *the brightness of an image formed by rays which make small angles with the axes, after refraction through any optical system, is equal to the brightness of the object.*

231. But this law must also hold, for any wide-angled aplanatic system, without the restriction that the rays shall make very small angles with the axis.

For if it did not hold, it would be possible to arrange an optical instrument in such a way that a person could make an object look brighter than it does to the naked eye, which contradicts all experiments made with different kinds of refracting media. And if this were possible for light, it would also hold for heat rays, whose laws of emission and refraction are the same as those of light; and this would contradict the law of the equality of radiation between bodies of equal temperatures. Hence we infer that the law of brightness

$$I : \mu^2 = I' : \mu'^2$$

holds for all aplanatic systems.

232. We are now able to extend the law connecting the angles of divergence of the initial and final pencils to wide-angled systems.

For consider, as before, the quantity of light sent out by an element of bright surface dS, placed perpendicular to the axis. The intensity of the emission of light in any direction varies as the cosine of the deviation of that direction from the axis. Hence the whole quantity of light sent out within a cone of semi-vertical angle α will be

$$L = IdS \int_0^\alpha \cos\theta \,.\, 2\pi \sin\theta \, d\theta,$$

or

$$L = \pi \, IdS \sin^2\alpha.$$

The corresponding formula for the emergent beam will be

$$L' = \pi \, I' \, dS' \sin^2\alpha'.$$

Equating these values, we find that

$$I\,dS\sin^2\alpha = I'\,dS'\sin^2\alpha'.$$

But
$$I : \mu^2 = I' : \mu'^2,$$
and also
$$dS : \beta^2 = dS' : \beta'^2;$$
hence the equation becomes

$$\mu\beta\sin\alpha = \mu'\beta'\sin\alpha'.$$

This expresses for wide-angled aplanatic systems, the relation between the angles of divergence of the initial and final pencils, which was found before for pencils of very small divergence. When the angles of divergence are very small, it is immaterial whether we write α, or $\tan\alpha$, for $\sin\alpha$.

This law was enunciated independently by Helmholtz and Professor Abbé of Jena in 1874.

233. It often happens in the case of instruments of high magnifying power, that the emergent pencil does not completely fill the pupil of the eye, and then the brightness of the image on the retina will be less than when the pupil is quite filled with the rays.

For let I_0 be the brightness when the eye is filled with the rays, and let λ be the distance of the image from the eye; then the section of the pencil by a plane coinciding with the pupil of the eye, will be $\pi\lambda^2\sin^2\alpha'$; and therefore if p is the radius of the pupil

$$I : I_0 = \pi\lambda^2\sin^2\alpha' : \pi p^2;$$

which gives
$$I = I_0\left(\frac{\lambda}{p}\right)^2\frac{\mu^2}{\mu'^2}\cdot\frac{\beta^2}{\beta'^2}\sin^2\alpha.$$

The last medium before the eye will be air, so that $\mu' = 1$, and β'/β is the magnifying power, which is denoted by m, and therefore

$$I = I_0\frac{\lambda^2}{p^2}\cdot\frac{\mu^2\sin^2\alpha}{m^2},$$

an equation which we shall find useful in dealing with optical instruments.

EXAMPLES.

1. A stereoscope is constructed of two glass prisms ($\mu = \frac{3}{2}$) with their edges coincident, and placed so that the faces of each are equally inclined to the plane on which the two pictures are placed, and at a distance of 6 in. The eyes of an observer are $2\frac{1}{2}$ in. apart; find their distance from the prism when the axes of the pencils from the middle points of the two pictures have minimum deviation and cross at the point half-way between them, the points being 4 in. apart. Show that the angles of the prisms must be nearly $\tan^{-1}\frac{3}{4}$.

2. Three convex lenses of focal lengths f_1, f_2, f_3 are separated by intervals a, b; find the magnifying power of the combination, and prove that it is independent of the position of the object if

$$(f_1 - a)(f_3 - b) + f_2(f_1 + f_3 - a - b) = 0.$$

3. The light after passing through an optical instrument symmetrical about an axis is reflected by a plane mirror perpendicular to its axis so as to pass through it again in the reverse direction; show that the compound instrument so formed is equivalent in every respect, if spherical aberration be neglected, to a simple spherical mirror, with its vertex in the position conjugate to the plane mirror and its centre of curvature at the corresponding principal focus.

4. If in any optical instrument formed of lenses and mirrors on the same axis, m is the magnifying power when the instrument is adjusted for an eye which sees clearly with the incident light parallel, and if the eye-glass (focal length f) is moved till the instrument is in adjustment for an eye whose distance of distinct vision is δ, show that the magnification is increased by mf/δ.

5. A cylindrical beam of light is incident on the object-glass of a telescope; find the distribution of the illumination in the focal plane and show from your result that the resolving power of a telescope increases with its aperture.

CHAPTER XI.

OPTICAL INSTRUMENTS.

234. WE have already treated the theory of vision through a single lens and its application to spectacles and reading-glasses. The next optical instrument in the order of simplicity is the simple microscope.

We have seen that when an object is placed at the focus of a convex lens, the rays of the several pencils will emerge parallel to each other, and therefore each pencil will be brought to a focus on the retina without effort; and in this position the angle under which it will appear to the eye is the angle it would subtend at a distance equal to the focal length of the lens. Consequently the image will be distinct and magnified. A lens of high power thus used is called a simple microscope.

If β denote the linear dimensions of the object, the tangent of the visual angle will be β/f, while the tangent of the angle under which it would be seen by the naked eye at the least distance of distinct vision is β/λ; the measure of the magnifying power is therefore λ/f. Single lenses answer very well so long as the focal length is not smaller than one inch; but when higher powers are required, combinations of more than one lens are preferable.

235. A form of simple magnifier, which possesses certain advantages over a double convex lens, is that commonly known as a "Coddington lens." The lens is spherical, but the rays are made to pass nearly through the centre of the lens. The first idea of it is due to Wollaston, who proposed to unite two hemispherical lenses by their plane sides, with a stop interposed, the central aperture of which should be equal to one-fifth of the focal

length. The same end was shown by Brewster to be attained more satisfactorily by grinding a deep groove round the equatorial part of a spherical lens, and filling it with something opaque. The great advantage of this lens is that the oblique pencils as well as the central pencils, pass normally into the lens, so that they are but little subject to defects of aberration.

The Stanhope lens consists of a cylinder of glass with its ends ground into spherical convex surfaces of unequal curvature; the length of the cylinder is so arranged that when the more convex end is turned towards the eye, objects placed *on* the other end shall be in the focus of the lens. This furnishes an easy way of mounting light objects for examination.

A modified form of the Stanhope lens, in which the further surface is plane, has been used extensively in France for the enlargement of minute pictures photographed on the plane surface; it is called a "Stanhoscope."

236. Wollaston was the first to use a combination of two lenses instead of a single lens; this combination is still known as *Wollaston's Doublet.* It was suggested by an inverted Huyghens' eye-piece, to be described presently. It consists of two plano-convex lenses whose focal lengths are in the proportion of 1 : 3, the plane surfaces being turned towards the object, and the lens of shorter focal length being placed next the object. The distance between the lenses can be adjusted to suit different eyes, but is usually $\frac{3}{2}$ of the shorter focal length.

Pritchard, who made doublets which magnified 200 to 300 diameters, performing excellently, made the distance between the lenses equal to the difference of their focal lengths, while the latter could vary in ratio from 1 : 3 to 1 : 6.

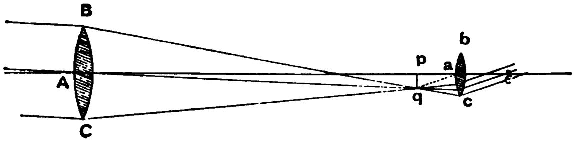
A better doublet was invented by Chevalier, who placed two plano-convex lenses of equal focal lengths but of different diameters, very close together, the larger being the nearer to the object, and between them he fixed a diaphragm. In this way he obtained more light and admitted a greater distance between the lens and the object.

Triplets have been constructed on the same principles. The combination with sufficient care of three plano-convex lenses gives even better results than doublets. They can be made com-

paratively free from aberration both spherical and chromatic. But they are so much inferior to the modern compound microscope that they are only used for rough observations or for dissecting.

237. The refracting telescopes and the compound microscope, in their simplest forms, consist of two lenses. The lens placed nearer to the object receives rays directly from the object and forms a real inverted image of the object; this lens is called the *object-glass*, or the *objective.* The inverted image is viewed by the eye through the other lens, which is called the *eye-glass* or *eye-piece;* this eye-glass alters the divergence of the small pencils which form the first image, so that they can be brought to a focus on the retina without effort, and increases the visual angle under which the image is seen. In general, an eye is accommodated for rays emerging parallel to each other; the eye-glass is therefore placed so that the first image is in the principal focus of this lens. In microscopes, however, where the magnifying power is very important, the instrument is arranged so that the final image is at a distance of about 10 inches from the eye; this distance is conventional, but is chosen once for all, so that the magnifying powers of different instruments may be compared under like circumstances.

The Astronomical Telescope.



238. The common Astronomical telescope, the construction of which was first explained by Kepler, consists primarily of two convex lenses fixed in a tube. In the figure, BAC is the lens which is turned towards the object, and it is therefore called the object-glass. This lens forms an inverted image pq, of the object, corresponding points of image and object lying on the same line through A, the centre of the object-glass. Bq, Aq, Cq are three rays diverging from any one point of the object which, after refraction by the object-

glass, are made to meet in q, the corresponding point of the image. These rays after crossing at q, fall upon the convex lens *bac*, called the *eye-glass*, and after refraction they are in general made to emerge parallel to each other. This will be effected by adjusting the position of the eye-glass, so that the image pq shall lie in its principal focus.

Let f, f' be the focal lengths of the object-glass and eye-glass, respectively. Then the angle qAp is the angle which the object subtends at the centre of the object-glass, and this will not differ sensibly from that subtended at the eye. By the naked eye, therefore, the object is seen under an angle whose tangent is $-\beta/f$, where β is the linear dimensions of the image. Also, the image pq will be seen through the lens at an angle whose tangent is β/f', wherever the eye be placed, supposing pq to be in the principal focus of the eye-glass. The magnifying power is therefore

$$m = -\frac{f}{f'}.$$

239. The field of view is defined by the axes of the extreme pencils which are transmitted by the eye-glass. It will therefore be the angle which the eye-glass subtends at the centre of the object-glass. Wherefore, if b' denote the semi-aperture of the eye-glass, and Θ half the field of view,

$$\Theta = \frac{b'}{f+f'}.$$

In order to take in the whole extent of this field the eye must be placed at the point in which the axes of the extreme pencils, diverging from the centre of the object-glass, meet the axis of the telescope on their final emergence. The place of the eye is therefore the focus conjugate to the centre of the object-glass as seen through the eye-glass. If x be the distance of this point outside the eye-glass,

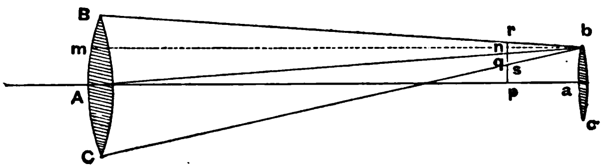
$$\frac{1}{x} + \frac{1}{f+f'} = \frac{1}{f'},$$

so that
$$x = \frac{f'}{f}(f+f').$$

In the construction of the instrument, the tube is prolonged to

the required distance and is there furnished with an eye-stop, and in looking through the instrument the eye is placed close to the end of the tube.

240. The field of view as defined by the *axes* of the extreme pencils is not the entire extent of the visible field, as determined by any rays whatever transmitted through both the lenses. For if we join the extremities and the centre of the object-glass to one extreme point of the eye-glass, and let the joining lines meet the common focal line in *rqs*, all the rays from the



object-glass which fall within *ps* strike the eye-glass; but only half the rays which meet at *q* are transmitted by the eye-glass, while only one ray of those meeting in *r* will meet the eye-glass. Thus all the field within *As* is seen by full pencils, while that between the lines *As* and *Aq* is seen by parts of pencils, the part exceeding half the pencil in each case; and the part of the field between *Aq* and *Ar* is seen by parts of pencils, the parts being less than half the pencil in each case. Let Θ', Θ'' be the values of half the bright field, and half the total visible field, respectively. Let the line *mnb* be drawn through the extremity of the eye-glass parallel to the axis of the telescope; then by similar triangles, $Cm : mb = sn : nb$. If we denote *ps* by y, and the semi-apertures of the lenses by b, b', respectively, this relation becomes,

$$\frac{b+b'}{f+f'} = \frac{b'-y}{f'},$$

which gives,

$$y = \frac{fb'-f'b}{f+f'}.$$

But $y = f\Theta'$; wherefore

$$\Theta' = \frac{fb'-f'b}{f(f+f')}.$$

To find the value of Θ'', we have only to change the sign of b; and therefore

$$\Theta''=\frac{fb'+f'b}{f(f+f')}.$$

If $b'/b=f'/f$, that is, if the apertures of the lenses are proportional to their focal lengths, Θ' vanishes; in this case the brightness of the field decreases from the centre to the circumference. If b'/b be less than f'/f, the value of Θ' becomes negative, and no part will be illuminated by full pencils.

The field as determined by the axes of extreme pencils, is limited by the line Aq, and therefore by elementary geometry, or by the values previously obtained,

$$\Theta' + \Theta'' = 2\Theta.$$

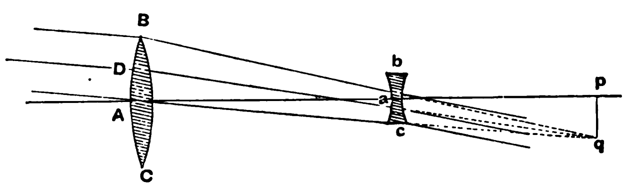
The field is limited practically to the bright field Θ', by means of a circular stop, which is placed at the principal focus of the object-glass, whose radius is

$$y=\frac{fb'-f'b}{f+f'}.$$

This will exclude the images of all points formed by partial pencils.

In an Astronomical telescope there is usually fixed a network of fine wires, vertical and horizontal, the plane of the wires being the focal plane of the object-glass. The image of the object given by the object-glass will then lie in the plane of the wires, and the image and the wires are viewed together through the eye-lens. By the aid of these wires the position of the image of any point can be accurately measured.

Galileo's Telescope.



241. This telescope, called after its inventor, Galileo, was the first whose construction was explained on theoretical principles.

It differs from the astronomical telescope chiefly in the form of its eye-glass, which is a double concave lens, and is placed between the object-glass and its principal focus. A pencil of light diverging from the object is brought to a focus by the object-glass; but before the rays reach this focus, some part of the pencil is caught by the eye-glass. In the annexed figure, BAC is the object-glass, bac the eye-glass, and pq is an inverted image of the object formed by the object-glass, corresponding points of the image and object lying on the same line through A, the centre of this lens. Bq, Aq, Dq, are three rays diverging from any point of the object, and after refraction they are made to converge to the point q, the corresponding point of the image. These rays fall upon the eye-glass and after refraction they are, in general, made to emerge parallel to each other. This will be effected when the eye-glass is so adjusted that the image pq is in its principal focus. When directed towards distant objects, pq is also in the principal focus of the object-glass, so that the distance between the lenses is then equal to the difference between the focal lengths of the two glasses.

Let β be the linear magnitude of the image pq, and f, f' the focal lengths of the object-glass and the eye-glass, respectively. Then the angle under which the object is seen by an eye placed at A is equal to the angle qAp, and this will not differ sensibly from the angle under which it will be seen by the eye in its proper position. The tangent of this angle is $-\beta/f$. Also the image pq will be seen through the lens under an angle whose tangent is $-\beta/f'$. The magnifying power is therefore

$$m = \frac{f}{f'}.$$

Thus the magnifying power is the same as in an astronomical telescope, the focal lengths of whose lenses are the same as in this instrument. The latter has the advantage of being shorter; for the distance between the lenses in this adjustment is equal to the difference between their focal lengths, whereas in the former it is equal to their sum.

A more important advantage which this instrument possesses is that through it objects are seen erect and not inverted, as in the Astronomical telescope. This is readily seen by following

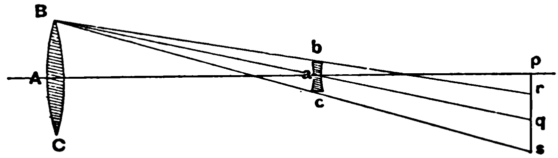
the course of the axes of extreme pencils as they diverge from the centre of the object-glass. When they meet the eye-glass they are made to diverge still more by it; and therefore the pencil flowing from the uppermost part of the object will proceed to the lower part of the retina, and *vice versa;* and therefore the object is seen in the same position as by the naked eye. On this account the instrument is convenient for viewing terrestrial objects. The ordinary opera-glass consists of a pair of Galileo's telescopes placed with their axes parallel, and arranged so that the distance between the lenses can be altered so as to adapt the telescopes for seeing objects at different distances.

242. The field of view in this instrument is very limited. For the axes of the pencils flowing from the several parts of the object, diverging from the centre of the object-glass, will diverge still more after refraction by the concave eye-glass, and therefore, for the most part, they will fall without the pupil of the eye and be lost. In order that the eye may receive as many as possible of these axes, it must be placed as near as possible to the point from which the axes diverge. This point, which is the eye-point, lies within the instrument, and therefore the eye cannot be placed at it, but will be placed close to the eye-glass. The effective aperture of the eye-glass is therefore reduced to that of the pupil, and it is useless to make the eye-glass of much greater aperture than the pupil. The value of the field of view as determined by the axes of the extreme pencils which strike the eye-lens, is therefore equal to the angle subtended by the pupil of the eye, at the centre of the object-glass. If b' be the semi-aperture of the pupil of the eye, and Θ denote half the field of view, then

$$\Theta = \frac{b'}{f-f'}.$$

But this is not the total visible field as seen by any rays whatever; this may be found as before. The aperture of the pupil being small compared with that of the object-glass, few, if any, whole pencils after refraction by the object-glass will fall within the pupil; it is therefore usual to regard those pencils which completely fill the pupil, as whole pencils, and those which

do not fill it as partial pencils; so that the field of view is limited by the object-glass.



We shall suppose that the semi-apertures of the lenses are b, b'; the latter being equal to the semi-aperture of the pupil. Let the extremity of the object-glass be joined in succession to the extremities and centre of the eye-glass, and let the joining lines meet the line pq in r, q, s respectively. Then pencils converging to any point within pr, will fill the eye-lens; the pencils converging to points in rq, will more than half-fill the lens, while pencils converging to points in qs will not half-fill the eye-glass. Hence if Θ' be half the bright field, and Θ'' half the whole visible field,

$$\Theta' = \frac{pr}{f}, \qquad \Theta'' = \frac{ps}{f}.$$

But, by similar triangles, it may be shown just as in the case of the Astronomical telescope that

$$pr = \frac{f'b - fb'}{f - f'};$$

so that

$$\Theta' = \frac{f'b - fb'}{f(f-f')}.$$

The value of Θ'' is obtained by changing the sign of b', and therefore

$$\Theta'' = \frac{f'b + fb'}{f(f-f')}.$$

In this telescope the principal focus of the object-glass is virtual, and therefore no stop and no network of fine wires for measuring, can be used.

Object-glasses.

243. We shall next apply the preceding theoretical considerations to the construction of good object-glasses.

One advantage of a telescope over the naked eye, in viewing a distant object, is the quantity of light which the instrument admits. The eye admits a small cone of rays issuing from each point of the object, just sufficient to fill the pupil; whereas a telescope admits a cone large enough to fill the whole object-glass. Thus a telescope enables us to see stars which are too faint to be perceived by the naked eye. The larger the aperture of the object-glass, the more light will be admitted. The first requisite of an object-glass is therefore a wide aperture.

We have seen that the brightness of an image is equal to that of the object; so that when the light from the image completely fills the pupil, just as light from the object does, they will appear of equal brightness. But when the magnifying power of the instrument is large, the emergent pencil never fills the pupil. When the telescope is directed towards a bright surface the emergent pencil fills the eye-ring. Let r be the radius of the eye-ring, and p the radius of the pupil; then, as has been remarked, r is usually smaller than p, and the apparent brightness will be less than the brightness of the object in the proportion of the areas of the eye-ring to that of the pupil. The brightness is therefore given by the equation

$$I = I_0\left(\frac{r}{p}\right)^2.$$

But if m be the magnifying power, $m = b/r$, where b is the semi-aperture of the object-glass. Hence

$$I = I_0\left(\frac{b}{mp}\right)^2.$$

Thus the brightness depends on the magnifying power and on the aperture of the object-glass; and if the magnifying power be large, the aperture of the object-glass must be large too, otherwise the brightness of the image will be impaired.

In Galileo's telescope the eye is placed close to the eye-lens, and the pupil is filled when points are seen by full pencils, and therefore the brightness of the image is very nearly equal to that of the object, and it does not depend on the aperture of the object-glass. But in this instrument the field of view depends on the aperture of the object-glass. This aperture, however, cannot be made very large, because the refraction through the lens is

excentrical, and if the aperture be large, the extreme pencils will be refracted at such a distance from the axis as to make the chromatic aberration considerable.

244. Object-glasses are usually made of two lenses, a convex lens of crown glass being combined with a concave lens of flint glass. The pencils of light are incident centrically on the first lens, and if there were an interval between the lenses, the incidence on the second lens would be excentrical; this would be disadvantageous, and the two lenses are placed close together.

We have therefore four quantities at our disposal, namely, the radii of curvature of the four surfaces of the two lenses.

The focal lengths of the two lenses are immediately determined by two essential conditions. These are, that the combination must have a given focal length, and must be achromatic. Let f, f' be the focal lengths of the lenses, and F the focal length of the combination. Then

$$\frac{1}{F} = \frac{1}{f} + \frac{1}{f'}.$$

Also the condition for achromatism is

$$\frac{\varpi}{f} + \frac{\varpi'}{f'} = 0.$$

These two equations determine f and f', so that no other condition can be satisfied which involves relations between the focal lengths.

We shall next consider errors due to aberration. The defects of an image formed by a single lens are

(i.) Distortion due to curvature of the image.

(ii.) Indistinctness due to obliquity in the more remote parts of the field.

(iii.) Indistinctness due to spherical aberration.

In the case we are considering there is no linear or angular distortion, because the incidence is centrical, and therefore the object and image are parallel sections of the same double cone.

It has been shown that if ρ, ρ' be the radii of curvature of the object and its image in the central parts of the field,

$$\frac{1}{\rho'} - \frac{1}{\rho} = -\left\{\frac{2k+1}{f} + \frac{2k'+1}{f'}\right\}.$$

We cannot therefore remove the defect of curvature, because it would involve a relation between the focal lengths of the constituent lenses.

Next, consider the effect of obliquity. After oblique centrical refraction at a lens, the pencils do not converge to a focus, but to two focal lines. To remove this defect it will be necessary to make the two focal lines after refraction through the two constituent lenses coincide. It has been shown that if v, v' be the distances of the focal lines from the centre of a single lens, for a pencil issuing from a point at a distance u,

$$\frac{1}{v}-\frac{1}{u}=\frac{1}{f}\left\{1+\phi^2\left(1+\frac{1}{2\mu}\right)\right\},$$

$$\frac{1}{v'}-\frac{1}{u}=\frac{1}{f}\left\{1+\frac{\phi^2}{2\mu}\right\},$$

where f is the focal length of the lens, and ϕ the obliquity, which is so small that powers of ϕ higher than the third may be neglected. To make the focal lines coincide after refraction through both the lenses, it will be necessary that

$$\phi^2\left\{\frac{1}{f}\left(1+\frac{1}{2\mu}\right)+\frac{1}{f'}\left(1+\frac{1}{2\mu'}\right)\right\}=\phi^2\left\{\frac{1}{2\mu f}+\frac{1}{2\mu' f'}\right\}.$$

This again involves a relation between the focal lengths which cannot be satisfied; so that except when $\phi=0$, we cannot ensure distinctness.

The defect due to spherical aberration is treated by making the aberration vanish for parallel rays. It has been shown how the aberration of the combination depends on two quantities ϵ, ϵ' which are independent of the focal lengths. To make the aberration for parallel rays vanish, it is necessary to impose a single relation between the quantities ϵ, ϵ', and we can still satisfy one more condition. Two courses are open to us. We may make the lenses fit together, so that they may be cemented together into one lens; or, as was shown in the chapter on aberration, we may make the aberration vanish not only for parallel rays, but for rays flowing from a point at a distance finite but considerable.

Eye-pieces.

245. In the Astronomical telescope instead of a single eye-glass it is usual to use a combination of two lenses separated by an interval. The introduction of a third lens between the object-glass and the eye-glass will increase the field of view of the instrument. For this reason it is usually called the *field-glass*.

The incidence of the pencils on the field-glass is not centrical, so that no advantage is gained by placing it close to the eye-glass. The two lenses of an eye-piece are therefore separated by an interval.

We have therefore five quantities at our disposal, namely, the four radii of curvature of the four surfaces of the lenses, and the distance between the lenses.

If f, f' be the focal lengths of the two lenses, a the distance between them, the focal length of the equivalent lens will be given by the equation

$$\frac{1}{F} = \frac{1}{f} + \frac{1}{f'} - \frac{a}{ff'}.$$

The focal length F of the combination will be a given quantity, so this is to be considered as one relation between the constants.

By far the most important defect of the image given by a single lens is that due to chromatic aberration. For a combination of two lenses separated by an interval, it is not possible to remove entirely the defects of this chromatic aberration. The defects of the image are two-fold, the coloured images are not in the same plane perpendicular to the axis of the telescope, and they are not of the same magnitude. Either of these defects can be removed but not both; and the first defect is of the less consequence and is therefore neglected. It is best to make the lenses of the same kind of glass, for then if the combination be achromatic as regards two colours, it will be perfectly achromatic, because there will be no irrationality of dispersion.

It has been shown in § 190 that the condition for this imperfect achromatism for two lenses of the same kind of glass is

$$a = \tfrac{1}{2}(f + f').$$

This is a second relation between the constants.

The errors of spherical aberration are more complicated than those which occur in the object-glass, because the pencils are incident on the eye-piece excentrically and with considerable obliquity.

The first defect is indistinctness, due to the spherical aberration of the more oblique pencils. When the central part of the image is at the proper distance from the eye-piece, the marginal parts will be too distant; and therefore if the central portion of the field be distinct, the marginal portions will be indistinct, and if the eye-piece be pushed inwards in order to see the marginal portions, the central portion will become indistinct.

The second defect is curvature of the image.

The third defect is linear and angular distortion. The axes of the extreme pencils proceeding from the centre of the object-glass will, by the spherical aberration of the eye-piece, meet the axis of the telescope at a nearer point than those of the central pencils. The ratio of the visual angles will therefore be greater in the extreme parts of the field than at its central parts; these extreme parts of the field will therefore appear unduly enlarged, and the object will appear distorted.

A fourth defect is due to the astigmatism of oblique pencils; the pencils after passing through the instrument will have two focal lines, and the image of a point formed on the retina by such a pencil will be in general an ellipse, becoming however sometimes a circle.

246. Without entering into details connected with these defects, it will be understood that the errors will, in general, be reduced by diminishing the aberrations of extreme pencils, and that if the forms of the lenses be given, this effect will be produced by increasing their number and dividing the refraction. The resulting aberration, other things being equal, will be least when the whole bending of the ray is equally divided among the lenses.

The condition for equal refraction is easily obtained. We shall confine our attention to two lenses. Let a ray, originally parallel to the axis meet the two lenses at distances y, y' from the axis. Then the deviations produced by the lenses are y/f, and y'/f', so that we must have $y/f = y'/f'$. But if θ be the inclination to the axis of the ray between the lenses

$$y' = y - a\theta, \text{ and } \theta = \frac{y}{f};$$

therefore $$y' = y\left(1 - \frac{a}{f}\right);$$

this gives $$f' = f\left(1 - \frac{a}{f}\right),$$

or finally, $$a = f - f'.$$

This condition, expressed in words, is that the interval between the lenses must be equal to the difference of their focal lengths. This is the principle on which Huyghens' eye-piece was constructed.

The preceding conditions only relate to the focal lengths and positions of the lenses, and are independent of their particular forms. The aberrations will depend largely on their forms; but the different defects previously mentioned in general require different and sometimes opposite forms for their correction. It is therefore necessary to sacrifice the perfection of the instrument in one respect to improve it in another which may be of more importance for the particular object for which it is intended. The theory of this part of the subject is however very troublesome, and it is but little attended to in practice. The lenses employed are almost invariably plano-convex or equi-convex lenses.

247. If we combine the condition of achromatism with the condition for equal refraction at the two lenses, we get the two equations

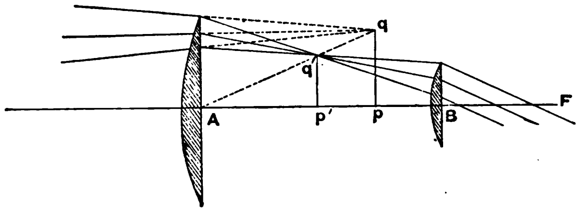
$$\left.\begin{array}{l} a = \frac{1}{2}(f + f') \\ a = f - f' \end{array}\right\}.$$

From these equations we deduce

$$f = 3f', \qquad a = 2f'.$$

The eye-piece will therefore consist of two lenses, the field-glass having a focal length equal to three times that of the eye-glass, and the distance between them equal to twice the focal length of the eye-glass. This is the construction of Huyghens' eye-piece, invented by him to diminish the effects of aberration, by making the deviations of the rays at the two lenses equal. It was afterwards

pointed out by Boscovich, that it also possessed the advantage of being achromatic.



The eye-piece is usually made with plano-convex lenses, the plane faces being next the eye. Rays proceeding from the object-glass would meet in q, qp being in the principal focal plane of the object-glass; the rays are caught by the field-glass before reaching q, and are brought to a focus at q', which is in the focus of the eye-glass, so that the rays will emerge parallel to each other. Let A, B be the centres of the lenses, AF the focal length of the lens A; then since $AF = 3f'$, $AB = 2f'$, the point F is also the principal focus of the lens B. Since $q'p'$ is in the focus of the lens B, $Bp' = \frac{1}{2}AB$. Also, since p, p' are conjugate foci with respect to the lens A,

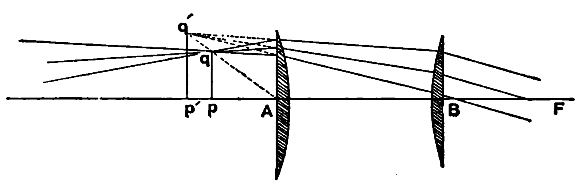
$$\frac{1}{Ap'} - \frac{1}{Ap} = \frac{1}{3f'}$$

and $Ap' = f'$; therefore $Ap = \frac{3}{2}f' = \frac{3}{4}AB$. Thus p is the middle point of AF.

Thus the field-glass must be placed between the object-glass and its principal focus, at a distance equal to half its own focal length from the latter.

This eye-piece cannot be used in telescopes where measurements by means of spider-lines or fine wires are to be made. For the principal focus of the object-glass is virtual. The wires could not be placed at the image qp, because there will be distortions in the image of the wires due to the eye-glass, while the image of the object will be distorted by excentrical refraction through both the field-glass and the eye-glass; so that the wires and the image will appear distorted in different degrees, and therefore the position of a point in the field would be estimated incorrectly by referring it to the wires. In all telescopes graduated by wires, for measurement, the field-glass must be beyond the principal focus of the object-glass; then the image and the wires if distorted at all are distorted equally, and therefore no error will result in the measurement.

248. In the common astronomical eye-piece, known as Ramsden's eye-piece, the two lenses are of equal focal length, and therefore the condition of achromatism requires that the distance between them should be equal to the focal length of either. But in this arrangement, the field-glass being exactly in the focus of the eye-glass, any dust which might happen to lie on it or any flaw in the glass would be magnified by the eye-glass and confuse the vision. The distance between the lenses is therefore made a little less than the focal length of either; and thus, though the eye-piece is not achromatic, the departure from perfect achromatism will not be great. The lenses are usually plano-convex lenses with their curved surfaces turned towards each other, and the interval between them two-thirds of the focal length of either.



Rays proceeding from the object-glass converge to a focus at q in the principal focal plane of the object-glass, and after crossing at q meet the field-glass. Their direction is then altered, so that they diverge from the point q', and this point is made to lie in the focal plane of the eye-glass, so that after refraction at the latter, the rays emerge parallel to each other. Let A, B be the centres of the two lenses, and let $AF = f$, the focal length of either, then $AB = \frac{2}{3}f$. Also since $q'p'$ is in the principal focus of the lens B, $Bp' = f$, so that $Ap' = \frac{1}{3}f$. Also p and p' are conjugate foci with respect to the lens A, and therefore

$$\frac{1}{Ap} - \frac{1}{Ap'} = \frac{1}{f};$$

and $Ap' = \frac{1}{3}f$, therefore $Ap = \frac{1}{4}f$.

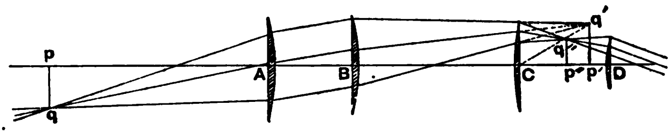
Thus the field-glass is placed beyond the focus of the object-glass at a distance from it equal to one-fourth of its own focal length.

The radii of the lenses are arranged so as to remedy as many of the defects of aberration as possible, and the indistinctness

arising from this cause in this eye-piece is much less than in any of the other ordinary constructions.

249. There is another eye-piece in common use, known as the erecting eye-piece; it is used for terrestrial objects. A terrestrial telescope differs from an astronomical telescope only in having an erecting eye-piece, instead of an ordinary eye-piece.

One form of erecting eye-piece is shown in the figure. *A* and



B are two convex lenses of equal focal length, placed at any distance from each other, *pq* is the image as formed by the object-glass. The lens *A* is adjusted so that *pq* lies in its principal focus; then it is easy to see that the lens *B* will form an image *p'q'*, of *pq* equal to it in magnitude but turned upside down, and the distance *Bp'* will also be equal to the focal length. Besides these two lenses there is an ordinary Huyghens' eye-piece which must be adjusted to the image *p'q'*, just as in the astronomical telescope. The distances between the four lenses are fixed; they are usually fitted into one tube, and adjustments for different distances are effected by pushing in or drawing out this eye-tube.

This eye-piece will not be achromatic, because the focal lengths of the two first lenses will not be the same for all colours.

250. The position of a compound eye-piece when arranged for distinct vision, and the magnifying power of the instrument, may be found by considering the images formed as the rays pass through the instrument.

We shall suppose the object to be very distant, and that the instrument is arranged so that the rays of the emergent pencils are parallel to each other, and therefore the first image will be in the principal focus of the object-glass, and the last image will be in the principal focus of the eye-lens. Let x, x' be the distances of these images in front of, and behind, the field-lens, respectively, and let β, β' be the linear magnitudes

of these images, and α, α' the initial and final inclinations of the axis of the extreme pencil. Then, if f, f', f'' be the focal lengths of the three lenses, and a, a' the intervals between them,

$$\left.\begin{aligned} a &= x + f \\ a' &= x' + f'' \end{aligned}\right\},$$

and
$$\frac{1}{x} + \frac{1}{x'} = \frac{1}{f'}.$$

The relation between the intervals a, a' is therefore

$$\frac{1}{a-f} + \frac{1}{a'-f''} = \frac{1}{f'}.$$

If we clear of fractions and add f'^2 to each side of this equation, it takes the form

$$(f + f' - a)(f' + f'' - a') = f'^2.$$

To find the inclinations of the initial and final pencils, the equations are

$$\alpha = \frac{\beta}{f},$$

$$\alpha' = -\frac{\beta'}{f''},$$

and also
$$\frac{\beta}{x} = -\frac{\beta'}{x'};$$

and therefore the magnifying power is

$$m = \frac{\alpha'}{\alpha} = -\frac{\beta'}{\beta}\frac{f}{f''},$$

or
$$m = \frac{x'}{x}\frac{f}{f''}.$$

But
$$\frac{x'}{x} = \frac{x'}{f'} - 1 = \frac{a' - f' - f''}{f'},$$

and therefore

$$m = -f\left\{\frac{1}{f'} + \frac{1}{f''} - \frac{a'}{f'f''}\right\}.$$

This formula might have been found directly by substituting for the two lenses of the eye-piece the equivalent lens, and then using the result already obtained for the magnifying power of the astronomical telescope consisting of two lenses.

In exactly the same way it may be shown that

$$\frac{1}{m} = -f''\left\{\frac{1}{f} + \frac{1}{f'} - \frac{a}{ff'}\right\}.$$

If we multiply these equations together we get the relation between the intervals already mentioned.

The object will be inverted, as in the ordinary astronomical telescope, unless a be greater than $f+f'$; that is, unless the distance between the first two lenses be greater than the sum of their focal lengths.

If the focal lengths of the lenses be given, and also the magnifying power, the intervals between the lenses are determined; for, from the preceding values of m, we get

$$a = f + f' + \frac{ff'}{f''m}, \quad a' = f' + f'' + \frac{f'f''m}{f}.$$

251. The field of view is determined in the same way as in the ordinary astronomical telescope, supposing it to be governed by the first two lenses. The aperture of the third lens will then be chosen so as to allow of all the rays to pass through. Thus if the field be determined by the axes of the extreme pencils, corresponding to the field of view Θ, let the semi-apertures of the field-lens and eye-lens be, respectively, b', b''. Then

$$\Theta = \frac{b'}{a}.$$

Also if α' be the inclination of the axis of the pencil after refraction at the field-lens, $\alpha' - \Theta$ will be the deviation produced by that lens, and therefore

$$\alpha' = \Theta - \frac{b'}{f'}.$$

Also $$b'' = b' + a'\alpha';$$

therefore $$b'' = b' + a'\Theta - \frac{a'b'}{f'}$$

$$= \Theta\left\{a + a' - \frac{aa'}{f'}\right\},$$

which determines the aperture of the eye-lens.

The aperture of the eye-lens corresponding to the greatest visible field and the bright field may be found in the same way.

In Galileo's telescope, the incidence on the eye-lens is centrical; the eye-lens used is therefore always a single concave lens, or an achromatised pair of lenses in contact.

Reflecting Telescopes.

252. If instead of a convex object-glass, a concave mirror be used to receive the rays proceeding from an object, an image of the object will be formed by the mirror, which, if the aperture be sufficiently large, may be viewed directly by means of an eye-piece placed in a suitable position, as in the case of the telescopes previously described. Such is the principle of Sir W. Herschel's telescope, which is the simplest of the reflecting telescopes.

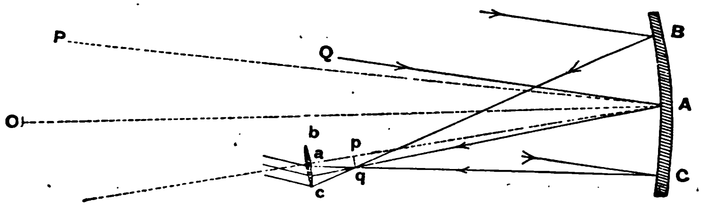
In order that the head of the observer may intercept as little light as possible, the axis of the mirror is slightly inclined to the axis of the tube in which it is fixed, and thus the image is thrown near the edge of the tube, where it is viewed through an eye-lens, or eye-piece, the observer having his back to the object and looking down into the tube. The obliquity of the incident pencil to the axis of the mirror will produce a slight distortion of the image, but the errors due to this cause are scarcely appreciable in the very large instruments to which this construction is alone applicable.

We shall suppose that the object is very distant, so that the image formed by the mirror will be in the principal focus of the mirror; and also that the instrument is to be adapted to the use of eyes with normal sight, so that the emergent rays must be parallel, and therefore the eye-lens must be placed in such a position that the first image may lie in its focal plane.

Now the angle which the object will subtend at the centre of the mirror, and therefore at the eye, will be equal to $-\beta/F$, where β is the linear magnitude of the image, and F the focal length of the mirror. And the angle under which the image will be seen by the eye will be β/f, f being the focal length of the eye-lens. The magnifying power is represented by the ratio of the latter to the former, and therefore

$$m = -\frac{F}{f}.$$

This instrument therefore gives an inverted image.



The arrangement of the mirror and eye-lens are shown in the figure. BAC is the large spherical reflector, AO being its axis, and O its centre; AP is the axis of the tube and Aa the axis of the eye-lens, and these two lines are equally inclined to AO, the axis of the mirror. Bq, Aq, Cq, are three rays which are brought to a focus at q by the large reflector; the rays afterwards meet the eye-lens and finally emerge parallel to each other. The focus q and the corresponding point of the object lie in the same line through O, the centre of the reflector.

253. The field of view in this telescope as determined by the axes of the extreme pencils is obtained by joining the extreme edges of the eye-lens to the centre of the large speculum. The distance between the lens and the centre is $F-f$, nearly; for $AO=2F$, and $Ap=F$ and the inclination of Ap to AO is very small. If therefore a denote the semi-aperture of the eye-lens, and Θ half the field of view,

$$\Theta = \frac{a}{F-f}.$$

The focal length of the eye-piece in these instruments is very small in comparison with that of the mirror so that *the field of view is very nearly equal to the angle subtended by the eye-lens at the vertex of the large reflector.*

The entire visible field is formed by joining the corresponding edges of the eye-lens and the reflector; the intercepted portion of the perpendicular erected at their common focus will be the linear magnitude which is illuminated by any ray whatever proceeding from the object, and the angle subtended by this perpendicular at the centre of the reflector will be the field of view. The determination is exactly the same as in the astronomical

telescope, and we may adopt the result previously obtained, namely,

$$\Theta' = \frac{Af + aF}{F(F+f)},$$

where A denotes the semi-aperture of the object-mirror, and Θ' half the extreme field. If we neglect f in comparison with F, and substitute m for F/f, this becomes

$$\Theta' = \frac{1}{F}\left\{a + \frac{A}{m}\right\}$$

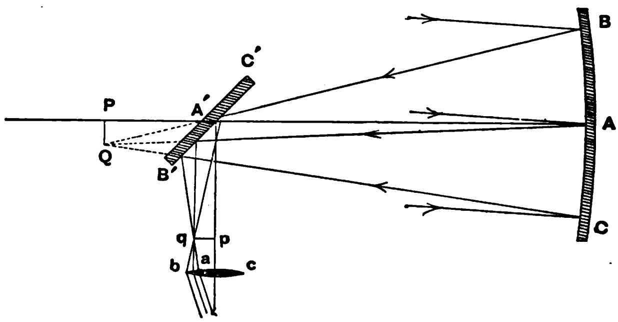
which, in instruments of large magnifying power, does not differ widely from the results previously obtained for the mean field.

Herschel's great telescope was constructed in 1789; it was 40 feet in length, and the great reflector was 50 inches in diameter. The quantity of light obtained by this instrument was so great as to enable its inventor to use eye-pieces of far shorter focal length than any previously used. Lord Rosse's telescope has a speculum of 53 feet focal length and 6 feet diameter.

Newton's Telescope.

254. The principle of the front view, as previously described, can only be used in instruments in which the aperture is very considerable, and to instruments of moderate aperture it is wholly inapplicable. In the telescope invented and constructed by Newton, the rays reflected by the object-reflector are received on a small plane mirror placed between the object-mirror and its principal focus. The plane of the mirror is inclined to the axis of the telescope at an angle of 45°, and the rays which tend to form an image in the principal focus of the object-reflector are reflected laterally and form an image near the side of the tube, equal and similar to the former, and similarly placed with regard to the plane mirror. This image, whose plane is parallel to the axis of the tube, is viewed through an eye-piece placed at the side of the instrument. Instead of a plane mirror, Newton used a rectangular isosceles prism of glass, through the sides of which the rays enter and emerge perpendicularly, being reflected totally at the hypotenuse. The reflexion at the hypotenuse being total, there is a much

smaller loss of light in the reflexion than in the reflexion at a metal speculum.



The arrangement of the mirrors and the eye-lens is shown in the figure. BAC is the object-mirror, $B'A'C'$ the plane mirror, and bac the eye-lens. Rays BQ, AQ, CQ are reflected by the large mirror to a focus Q, where PQ is the principal focal plane of the reflector. But before they reach Q they are reflected by the small plane mirror and meet in q; after crossing at q they strike the eye-lens and emerge parallel to each other. The point Q and the corresponding point of the object lie on a line through the centre of the large reflector; also the image PQ and the second image qp are symmetrically placed with regard to the mirror $B'A'C'$, and qp is equal in magnitude to QP.

If F, f denote the focal lengths of the object-mirror and the eye-lens, and e, e' denote their distances from the centre of the plane mirror, then in the figure,

$$\left.\begin{aligned} A'P &= F - e \\ A'p &= e' - f \end{aligned}\right\},$$

since the first image is in the principal focal plane of the large mirror, and the last in that of the eye-lens. Hence, since $A'P = Ap$, we get

$$e + e' = F + f.$$

This is the condition of distinct vision with parallel rays.

The magnifying power may be found just as in the case of Herschel's telescope; the value of it is, as before,

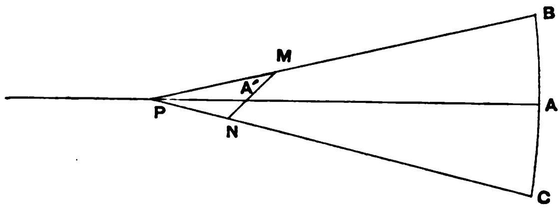
$$m = \frac{F}{f}.$$

255. The small mirror must be large enough to receive the whole of the principal pencil, or the cone of rays meeting in the principal focus of the object-mirror, but it must not be made larger than necessary, or otherwise the brightness of the central part of the field will be impaired. The mirror will therefore be a section of the full cone of rays converging from the object-mirror to its principal focus, made by a plane at an angle of 45° to the axis; it will therefore be in the form of an ellipse.

Let the semi-vertical angle of the cone be θ, then

$$\tan\theta = \frac{A}{F},$$

where A denotes the semi-aperture of the reflector.



Suppose a section of the cone by a plane through the axis perpendicular to the plane mirror to be represented in the figure, MN being the section of the plane mirror; then MN will be the major axis of the ellipse. Denote the two portions of MN, as divided in the centre A', by x and x'. Then if $A'P$ be denoted by d,

$$x = \frac{d\sin\theta}{\sin(45^\circ - \theta)} = \frac{d\sqrt{2}\tan\theta}{1-\tan\theta} = \frac{Ad\sqrt{2}}{F-A},$$

$$x' = \frac{d\sin\theta}{\sin(45^\circ + \theta)} = \frac{d\sqrt{2}\tan\theta}{1+\tan\theta} = \frac{Ad\sqrt{2}}{F+A},$$

and therefore if a, b denote the semi-axes of the ellipse,

$$a = \tfrac{1}{2}(x+x') = \frac{AFd\sqrt{2}}{F^2-A^2}.$$

Let y denote the breadth of the section perpendicular to that represented in the figure, at A'; then by properties of the ellipse,

$$\frac{b^2}{a^2} = \frac{y^2}{xx'}.$$

But y is the radius of the circular section of the cone through the point A, so that $y = Ad/F$; and therefore

$$\frac{b^2}{a^2} = \frac{F^2 - A^2}{2F^2}.$$

If we give a its value, the corresponding value of b becomes

$$b = \frac{Ad}{\sqrt{F^2 - A^2}}.$$

The aperture of the object-mirror will be small compared with its focal length, and therefore A^2 may be neglected in comparison with F^2. The approximate values of a and b will therefore be

$$a = \frac{Ad\sqrt{2}}{F},$$

$$b = \frac{Ad}{F},$$

which are in the ratio of $\sqrt{2}$ to 1.

256. The extreme field of view in Newton's telescope is determined by joining the adjacent extremities of the eye-lens and the plane mirror. The intercepted portion of the perpendicular qp, raised at the principal focus of this lens, will give the whole extent of the image illuminated by any rays whatever, proceeding from the object. The field of view will be the angle subtended at the centre of the object-mirror by the corresponding image PQ. Let y be the magnitude of this image, a the semi-aperture of the eye-lens, A' the perpendicular distance of the extremity B' of the small mirror from the line $A'a$. Then, by similar triangles,

$$\frac{a - y}{f} = \frac{y - A'}{d - A'}$$

where d denotes the distance $A'P$, or $A'p$. Neglecting A' in comparison with d, in this result, we get

$$y = \frac{ad + A'f}{d + f};$$

and if Θ be half the visible field,

$$\Theta = \frac{1}{F}\frac{ad + A'f}{d + f}.$$

From the preceding investigations, $A' = Ad/F$, very nearly. If we substitute for A' this value, and neglect f in comparison with d, we get

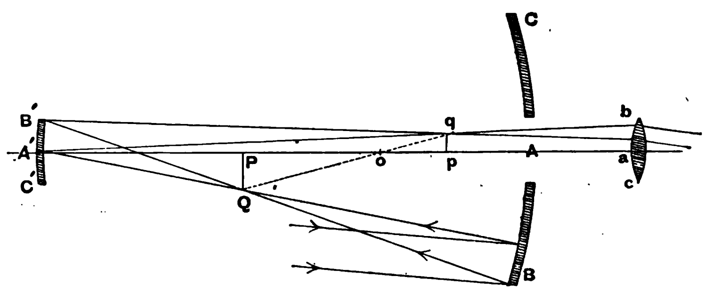
$$y = a + \frac{A}{m},$$

and $$\Theta = \frac{1}{F}\left(a + \frac{A}{m}\right), \text{ as in Herschel's telescope.}$$

Gregory's Telescope.

257. The invention of the reflecting telescope is generally ascribed to James Gregory, who described the instrument now called by his name, in his *Optica promota*, published in the year 1663.

Gregory's telescope consists of two concave mirrors placed along the same axis with their concavities facing each other, and at an interval a little greater than the sum of their focal lengths. In the vertex of the larger, or object-mirror, is a circular aperture, to which is attached the tube containing the eye-lens. When the axis is directed to a distant object an image is formed at the principal focus of the object-mirror. The rays diverging from this image are incident upon the smaller mirror, and by reflexion a second image is formed near the vertex of the large mirror, and this image is viewed through the eye-lens, placed at a distance from it equal to its own focal length.



The arrangement of the mirrors and the lens is shown in the figure, in which AB is the object-mirror, $A'B'$ the smaller mirror, and ab the eye-glass. Rays proceeding from a point of the object are reflected at the large mirror and are brought to a focus in Q,

where PQ is in the principal focal plane; also Q and the corresponding point of the object lie on the same line through the centre of the large mirror. After passing Q, the rays diverge and are incident on the small mirror, and are brought to a focus at q; as before, the points Q, q lie on the same line through the centre of the small mirror. The eye-lens is placed in such a position that qp is in its principal focal plane, and therefore the rays of the pencil after passing through the eye-lens emerge parallel to each other.

In the original description of the instrument the large reflector was a paraboloid of revolution, and the smaller, a prolate spheroid whose foci are at P and p, the positions of the two images. With reflectors formed of these surfaces, there would be no aberration for rays in the centre of the field. It was for some time deemed hopeless to prepare mirrors having these forms, and the instrument was never constructed till after that of Newton.

Gregory's telescope is generally preferred to Newton's. Its superiority seems to arise from the fact that the two specula may be matched and their irregularities of form made to counteract each other; whereas in Newton's telescope there is nothing to compensate any defect in the form of the object-mirror, and experience shows that such mirrors can seldom be made truly spherical.

258. Let F, F' and f be the focal lengths of the two mirrors and the eye-lens, respectively; e and e' the distances of the object-mirror and the eye-lens from the smaller mirror, and x, x' the distances of the two images from the same mirror. Then when the instrument is arranged for distant objects so as to suit normal eyes,

$$\left.\begin{aligned} x &= e - F \\ x' &= e' - f \end{aligned}\right\}.$$

But x, x' are conjugate focal distances with respect to the smaller mirror, and therefore,

$$\frac{1}{x} + \frac{1}{x'} = \frac{1}{F'}.$$

If we eliminate x, x', we get the equation

$$\frac{1}{e-F} + \frac{1}{e'-f} = \frac{1}{F'}.$$

This is the equation of condition for distinct vision in Gregory's telescope. The equation is similar to that previously obtained for a refracting telescope with three lenses.

The eye-glass is usually fixed in position and the adjustment to distinct vision effected by moving the smaller mirror by a fine screw.

Let β, β' be the linear magnitudes of the first and second images. Then the angle subtended by the object to the eye is equal to $-\beta/F$, and the angle under which the last image is seen is equal to β'/f. The magnifying power of the instrument is therefore

$$m = -\frac{\beta'}{\beta}\frac{F}{f}.$$

But in § 40 it was shown that in reflexion at a spherical surface the relation between the linear magnitudes of an object and its image is expressed in the equation

$$\frac{\beta}{x} + \frac{\beta'}{x'} = 0,$$

and therefore

$$m = \frac{F}{f}\frac{x'}{x}.$$

Also

$$\frac{x'}{x} = \frac{x'}{F'} - 1 = \frac{e' - f}{F'} - 1,$$

and therefore

$$m = -F\left\{\frac{1}{f} + \frac{1}{F'} - \frac{e'}{fF'}\right\}.$$

Similarly it may be shown that

$$\frac{1}{m} = -f\left\{\frac{1}{F} + \frac{1}{F'} - \frac{e}{FF'}\right\}.$$

The values of e, e' as obtained from these equations are

$$e = F + F' + \frac{FF'}{fm}, \qquad e' = F' + f + \frac{F'fm}{F},$$

which determine the intervals between the mirrors and the lens when the focal lengths and the magnifying power are given.

The first form for m gives a simple approximate value of the magnifying power. For, since the first image is nearly at the principal focus of the smaller mirror, and the second nearly at the

vertex of the greater, $e' - F' - f = F$, very nearly, and therefore we get

$$m = \frac{F^2}{F'f}.$$

The second image being inverted with respect to the first, and the first with respect to the object, the image as seen through the telescope is erect.

259. The smaller mirror must be of such dimensions as to receive the whole cone of rays converging from the large mirror to its principal focus; if it be greater than this, it will intercept more than is necessary of the incident pencil. The aperture of the smaller mirror is therefore determined by the equation

$$A' = A \cdot \frac{x}{F} = A\left(\frac{e}{F} - 1\right).$$

The aperture in the vertex of the object-mirror must not exceed the aperture of the smaller mirror, for otherwise some of the incident light would fall directly upon the eye-lens; it is usual therefore to make the aperture equal to that of the small mirror, in order that the aperture of the eye-glass, and therefore the field of view, may be as large as possible.

260. The extreme field of view in Gregory's telescope is found by joining the corresponding extremities of the small mirror and the eye-glass by the line $B'b$; the intercepted portion of the image qp, will determine the field. If a be the semi-aperture of the eye-glass, and β' half the linear magnitude of the second image thus determined, it will be seen that

$$\frac{a - \beta'}{f} = \frac{\beta' - A'}{x'},$$

and therefore
$$\beta' = \frac{ax' + A'f}{x' + f}$$
$$= a + A'\frac{f}{x'},$$

nearly, since f is small in comparison with x'. It has already been seen that $A' = Ax/F$, and therefore

$$\beta' = a + \frac{A}{m},$$

where m is the magnifying power.

Also, using the same value of m as before,

$$\frac{\beta'}{f} = m \cdot \frac{\beta}{F} = m\Theta,$$

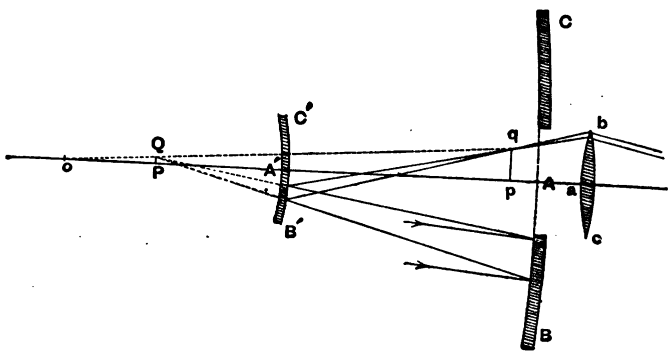
where Θ denotes half the field of view; and therefore

$$\Theta = \frac{1}{mf}\left(a + \frac{A}{m}\right).$$

The second term within the brackets is, in general, small compared with the first, and therefore the field of view is approximately

$$\Theta = \frac{a}{mf}.$$

261. Another reflecting telescope was invented some years after Gregory's and Newton's telescopes by a Frenchman named Cassegrain, probably without any knowledge of what had been done in England. Cassegrain's telescope only differs from Gregory's in having its small mirror convex instead of concave, and placed between the large mirror and its principal focus. The arrangement of the mirrors and images is shown in the figure.



The investigations to find the position of the mirrors and lenses so as to admit of distinct vision, and the magnifying power and the field of view in Gregory's telescope are all applicable to Cassegrain's; we have only to change the sign of F', the focal length of the small mirror, throughout.

The image will appear inverted, just as in the astronomical telescope.

262. The Cassegrain construction has not been much used, but it possesses certain advantages over Gregory's. For the same magnifying power, Cassegrain's telescope is the shorter of the two; but the great advantage of Cassegrain's telescope arises from the fact that the spherical aberrations of the two mirrors lie in opposite directions, so that they tend to neutralise each other, whereas in Gregory's telescope the two spherical aberrations are added together. It has been seen that when a pencil is directly reflected at a spherical surface, the caustic points in the direction tending from the mirror towards the centre. If we refer to the diagram representing Gregory's telescope, it will be seen that the first image PQ is displaced by the spherical aberration of the large mirror so as to make it fall nearer to this mirror than before. The distance x will therefore receive an increment ∂x in the positive sense, where

$$\partial x = \frac{y^2}{8F},$$

y being the semi-aperture of the large mirror.

Independently of the spherical aberration of the small mirror this will produce a change in x'; for

$$\frac{1}{x} + \frac{1}{x'} = \frac{1}{F'},$$

and therefore $$\frac{\partial x}{x^2} + \frac{\partial x'}{x'^2} = 0.$$

Thus we find $$\partial x' = -\frac{x'^2}{x^2}\frac{y^2}{8F}.$$

Besides this displacement of the second image pq, there will be another due to the spherical aberration of the small mirror. If we denote the semi-aperture of this mirror by y', and call O' the centre of the mirror, the change in x' due to the spherical aberration of the mirror will be

$$\Delta x' = -\frac{y'^2}{8F'^3}(O'p)^2.$$

The whole change in x' will be the sum of $\partial x'$ and $\Delta x'$; in Gregory's telescope those quantities have the same sign, but in Cassegrain's telescope we must change the sign of F', so that $\partial x'$ and $\Delta' x$ have opposite signs.

The Compound Microscope.

263. In its simplest form the compound microscope, like the astronomical telescope, consists of two lenses, an object-glass or objective, as it is usually called, and an eye-glass or eye-piece. The objective has a very short focal length, and the object is placed at a distance from it slightly greater than the focal length; the objective then forms a real inverted image of the object, which is viewed through the eye-piece.

The objective is usually made up of a system of lenses, designed to diminish chromatic and spherical aberration. Very generally, there are three doublets, each consisting of a double-convex lens of crown-glass cemented to a plano-concave lens of flint, arranged to be achromatic for centrical pencils; these doublets are placed with their plane faces towards the incident light, the lens of shortest focal length being next the object, and their apertures increasing from the first outwards. In this way the apertures can be chosen that a pencil filling the first lens will just fill the other lenses in succession, so that diaphragms are unnecessary; this is a great advantage because diaphragms will always introduce diffraction fringes which interfere with the definition in the outer parts of the field.

264. The preceding theory of aberration as given in Chapter VII. is quite useless when applied to microscopic objectives. For in this theory the inclinations of the rays to the axis of the instrument were supposed to be so small that the cubes of their circular measures might be neglected. But in microscopic objectives the incident cone of rays has a semi-angle in air up to 60° or 70°. Good objectives can be made to be used in air, which are nearly perfectly free from spherical aberration, with an aperture angle up to 105°, or 110°.

The explanation of this remarkable fact was given by Mr. J. J. Lister in a paper published in the Philosophical Transactions of 1830; and the conclusions there arrived at served as the point of departure for subsequent improvements. He found that for an achromatic object-glass of the type described above there are two points on the axis of the lens for which the lens is aplanatic for

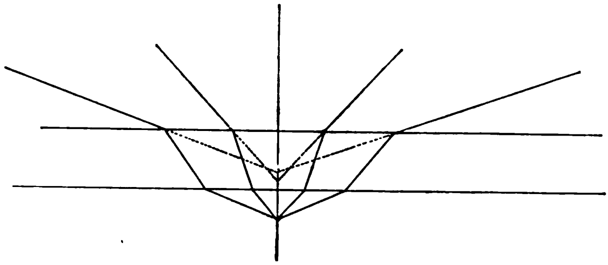
moderate apertures; the further aplanatic focus lies a little beyond the principal focus on the plane side, and the other lies nearer the lens. He also found that when the radiant point lies between the aplanatic foci, the spherical aberration is over-corrected, that is, has a sign opposite to that which might have been expected from a collective lens; but when the radiant point lies beyond these two foci either way, the spherical aberration is under-corrected.

The spherical aberration of a combination of two such achromatic lenses may therefore be neutralised by altering the positions of the lenses. For if we allow the first lens to receive rays from an object lying in its nearer aplanatic focus, and place the second lens so that the pencil incident on it comes from its further aplanatic focus, the aberration will be nearly perfectly corrected. If the lenses be brought a little nearer the combination is over-corrected, while their separation will produce under-correction.

The principle of the construction of the objectives with the wide apertures now in use, is to compensate the irremoveable errors of the lenses of higher powers by intentionally introducing opposite aberrations of the lower ones. For the very large angles of modern objectives the usual type is to have a single nearly hemi-spherical front lens, combined with a strongly over-corrected system of lenses. The latter system sometimes consists of two doublets, or a doublet and a triplet, and sometimes even of a front triplet, a middle doublet and a back triplet. But to make these complex systems efficient, extraordinary care has to be exercised in their grinding and centreing.

The corrections for colour are made in a similar manner. In addition to secondary spectra, there exists in microscopes a second defect, pointed out by Professor Abbé, and called by him *chromatic difference of the spherical aberrations.* This term he uses to denote the fact that when the lenses are designed and arranged to correct the spherical aberration for rays of mean refractive index, there will be a slight residual spherical aberration for all rays not of this refractive index; the lenses will be spherically under-corrected for red rays and spherically over-corrected for violet rays. In a paper on this subject he shows how this defect can be approximately removed by suitably choosing the distances between the lenses.

265. When wide-angled instruments are used, it becomes necessary to consider the aberration due to their cover-glass. When



a ray of light passes through a thin plate, its direction on emergence is the same as on incidence, but the ray appears to be proceeding from a different point; a pencil diverging from a point and passing through the plate, will at emergence, no longer be diverging from a point, but from a series of points along the axis of the system. The error introduced is of exactly the same kind as the spherical aberration of a lens, and may be corrected by adjusting the lenses. It is usual to make part of the objective system moveable relatively to the other part of the instrument, so that the parts may be arranged to suit any thickness of cover-glass. This is effected by means of a screw collar, which is graduated for different thicknesses of cover-glass.

266. In the best modern microscopes, a drop of fluid is introduced between the cover-glass and the front face of the objective. The fluid used at first was water; but the advantages of the water-immersion are obtainable with greater completeness by using *homogeneous immersion.* Professor Abbé after a long series of experiments found that oil of cedar-wood very nearly corresponds to glass in its refractive index and dispersive powers, so that by using it, rays issuing from the cover-glass at any angle enter the front lens without any change due to refraction or any loss from reflexion; even the most oblique rays proceed in their undeflected course until they meet the back surface of the front lens. By this means a much larger angular aperture can be obtained than by any dry-objective.

But this is not the only advantage to be derived from the immersion system. It appears from the investigation of the

brightness of images, that the brightness of the image is given by the formula

$$I = I_0 \frac{\lambda^2}{p^2} \cdot \frac{\mu^2 \sin^2 \alpha}{m^2},$$

where μ is the refractive index of the medium in which the object lies. This shows the advantage of having μ large. It has indeed been established by Clausius for heat as well as for light, that the radiation of a body into a medium is proportional to the square of the refractive index. Thus more rays are received by the objective with the immersion system than with the dry system. The physical explanation of this is shown by Prof. Abbé to depend on radiation by *diffracted* light. The loss of rays by the dry system cannot be compensated for by increase of illumination, for the rays which are lost are different rays physically, from those obtained by any illumination, however intense, in a medium like air.

267. The eye-pieces used in microscopes are of the same kind as for telescopes; usually Huyghens' eye-pieces are used, except for special purposes where measurements are necessary, in which cases Ramsden's eye-pieces are used. With each microscope there are generally supplied three eye-pieces of different powers; the eye-piece of high power is often called *deep*.

Referring again to the expression for the brightness of the image, we find that when the magnifying power is very large, the brightness diminishes and, other things being equal, the brightness varies inversely as the square of the magnifying power. It is therefore necessary to supply light artificially. This is done by concentrating light on the under side of the object by a reflector and lens; the light shines through the object and passes into the instrument illuminating the details of the object. Different forms of illuminators are used; but for descriptions we must refer to books on the microscope.

Magnifying power of the Microscope.

268. To obtain a correct measure of the magnifying power of an instrument, we must compare the magnitudes of the retinal images, first when the eye is used in combination with

the instrument, and secondly when the eye is used alone. But before this comparison can be definite, we must say where the object and the image formed by the lens system must be placed, in order that the retinal images formed may be fit for the determination of the magnifying power. To make this comparison correct, the eye, and the combination of the eye and instrument, must be compared as much as possible under analogous circumstances; this may be realised by comparing them while working as favourably as possible, that is, when they give the largest possible images on the retina. For the eye alone, the object must therefore be placed at the nearest point for distinct vision. But the smallest distance for distinct vision is very different for different persons; whereas the magnifying power ought to give an idea of the amplification of the instrument for the eye in general. It has therefore been agreed to place the object at a distance conventionally fixed, a distance not too great for the retinal images to be near their greatest dimensions, and which is large enough for the great majority of eyes to remain accommodated for it during a long time. The distance chosen is 10 inches, and is generally called the "distance of distinct vision." The phrase is not a happy one, for at every distance at which an eye can accommodate itself, it sees equally distinctly. The distance chosen for the position of the image formed by the lens system is the same; for then the retinal images will be proportional to the linear magnitudes of the object and image themselves.

Let x, x' be the distances of the object and final image from the first and second principal points of the system, and f the principal focal length. Then

$$\frac{1}{x}+\frac{1}{x'}=\frac{1}{f}.$$

Let β, β' be the linear magnitudes of the object and image, then

$$\frac{\beta}{x}=-\frac{\beta'}{x'}.$$

Also if the distance of the eye from the second principal point be ξ, the angle under which the image will be seen is

given by the equation

$$\tan\theta = \frac{\beta'}{\xi - x'} = \frac{\beta}{x\left(1 - \frac{\xi}{x'}\right)}.$$

Now f is very small, and x' will be negative, and the eye will be not far from the principal point, so that ξ/x' may be neglected, and

$$\tan\theta = \frac{\beta}{x}, \text{ very nearly.}$$

Also supposing the image at the conventional image distance λ, $x' = -\lambda$, and therefore

$$\tan\theta = \beta\left(\frac{1}{\lambda} + \frac{1}{f}\right).$$

When the object is viewed by the eye at the distance λ, it is seen under an angle θ_0, where

$$\tan\theta_0 = \frac{\beta}{\lambda};$$

and therefore the measure of the magnifying power will be

$$m = 1 + \frac{\lambda}{f}.$$

In general f is very small compared to λ, and therefore the magnifying power is

$$m = \frac{\lambda}{f}.$$

On the measure of the aperture of the microscope.

269. It has been shown that the brightness of an image given by a microscope is determined by the formula

$$I = I_0\frac{\lambda^2}{p^2}\cdot\frac{\mu^2\sin^2\alpha}{m^2},$$

where λ is the conventional image distance, p the radius of the pupil of the eye, m the magnifying power, and α the divergence of the cone of rays proceeding from the object in a medium whose refractive index is μ. Thus for an instrument of given magnifying power,

$$I \propto (\mu\sin\alpha)^2,$$

and accordingly, $\mu\sin\alpha$ may be taken to be the numerical measure of the aperture.

This measure of the aperture may be expressed in terms of the focal length of the objective, and diameter of the pencil passing through it. The diameter of the pencil as it passes through the objective varies from the first to the last. We shall suppose that the diameter is taken at the back surface of the objective as the pencil emerges from it. This will be so close to the second principal focus of the objective in microscopic objectives of the ordinary type of construction, that the difference in the distance may be disregarded. We shall therefore suppose that b is the semi-diameter of the pencil at the second focal plane of the objective, and that f is the focal length of the objective. Let u' be the distance of the image from the second principal focus; then, using the ordinary notation

$$\frac{\beta'}{\beta} = -\frac{u'}{f}.$$

Also by Helmholtz' theorem, we have

$$\mu\beta \sin\alpha = \mu'\beta' \sin\alpha',$$

and therefore

$$\mu \sin\alpha = \mu' \frac{\beta'}{\beta} \sin\alpha'$$

$$= -\frac{\mu'}{f} u' \sin\alpha'.$$

The angle α' is always very small in microscopes, never exceeding a few degrees, and therefore $u' \sin\alpha'$ will not differ sensibly from $u' \tan\alpha'$. But $b = -u' \tan\alpha'$, and therefore

$$\mu \sin\alpha = \frac{\mu' b}{f}.$$

The last image is always formed in air, so that $\mu' = 1$, and therefore finally

$$\mu \sin\alpha = \frac{b}{f}.$$

270. This numerical measure of the aperture may be justified by general reasoning. Other things being equal, it is clear, that the numerical measure of the aperture ought to vary as the diameter of the pencil. Next, suppose we have objectives of the same diameter of opening, but of different focal lengths. Imagine rays traced

backwards through the two objectives in succession from the same object. The incident rays are nearly parallel, and since the openings of the objectives are the same, they will admit backwards the same number of rays. But these rays will be concentrated to a smaller area by the lens of shorter focal length than by the other, the linear dimensions of the areas varying as the focal lengths, but their brightness being the same. Reverting to the original arrangement of the instrument, the objective of shorter focal length will admit the same number of rays from the smaller area, as the other will admit from the larger area. The real aperture of the former is therefore greater than the other in the inverse ratio of their focal lengths.

The value b/f is independent of the medium in which the object is placed; it is the same for air, water, balsam or any other immersion system. A numerical aperture *unity*, would correspond to an incident cone of rays in air whose vertical angle is 180°, while with homogeneous immersion the same aperture would correspond to a cone of angle $82^\circ\ 17'$; and with modern objectives the apertures reach 1·40 and sometimes more than this.

271. The magnifying power of an objective may be measured for a definite position of the image, by projecting the image of a stage-micrometer upon an eye-piece micrometer. And then we can find the numerical aperture of the objective by means of the formula

$$\mu \sin \alpha = \frac{mb}{u'}.$$

For an auxiliary microscope may be focused to the focal plane, and the linear diameter $2b$ of the emergent pencil measured there; then we have only to measure u', the distance of the focal plane from the image to which m refers, and we have the means of finding the value of $\mu \sin \alpha$.

Conversely, if we know the numerical aperture, the focal length of the object-glass may easily be measured; for using the formula

$$\mu \sin \alpha = \frac{b}{f},$$

we have only to measure micrometrically the diameter $2b$ of the pencil as it emerges at the principal focal plane.

For further details of the construction and theory of the

microscope, we may refer to "Nägeli u. Schwendener, Das Mikroscop," and the various papers of Professor Abbé of Jena, scattered through the *Journal of the Microscopical Society*, and other periodicals. It is to Prof. Abbé that a great many of the most recent developments of the theory, and improvements in the construction of microscopes are due.

It may be added that Prof. Abbé has recently been applying his new glasses to the construction of new objectives, which are now being made by the famous firm of Zeiss of Jena. To his new achromatic lenses Professor Abbé gives the name of *apochromatic* lenses. He claims for them great superiority in the finer qualities of definition, the new dry apochromatic lens giving an image as good as that of an ordinary achromatic water-immersion objective. He also claims that the more perfect corrections permit equal magnification to be obtained by using a longer-focus objective with an eye-piece of higher power than hitherto has been usual, thus obviating some of the difficulties attending very short-focus objectives. Moreover the foci for visible and for photographic purposes are identical. Special compensating eye-pieces have also been devised for use with the new apochromatic objectives.

For further information about these recent improvements we refer to a paper by Prof. Abbé published in the *Journal of the Royal Microscopical Society*, February, 1887.

EXAMPLES.

1. A Wollaston's doublet is formed of two lenses of focal lengths f, and $3f$, respectively, and is in adjustment for viewing a small flat uncovered object; show that if a plate of glass whose thickness is $f/10$ be laid on the object, the instrument may be readjusted without altering the position of the lower lens, by increasing the distance between the lenses by $2(\mu-1)f/(6\mu-1)$.

2. In an astronomical telescope, in which the focal lengths of the object-glass and eye-glass are f, f' and their semi-diameters b, b', respectively, show that for a person who can see distinctly at a distance a, the diameter of the stop should be

$$\frac{a(fb'-f'b)+ff'b'}{ff'+a(f+f')}.$$

3. A Galileo's and a common telescope have the same object-glass, and their eye-glasses have equal focal lengths, also the uniformly bright field is of the same extent in both; prove that the diameter of the stop in the common telescope should be half the difference of the breadths of the eye-glasses.

4. If the final image formed by the object-glass be at a distance from the eye-piece equal to the least distance of distinct vision, show that specks on the object-glass cannot be distinctly seen when the eye is close to the eye-piece, and find how far off the eye must be in order to see them distinctly.

5. Find the radius of the stop in an astronomical telescope for an observer who sees objects clearly at a distance λ; show that the stop may be greater than for a person seeing distinctly with parallel rays by $(b+b')f'^2/\lambda f$, if the square of f'/f be neglected, f and f', b, b' being the focal lengths and the semi-apertures of the two lenses, respectively.

6. In Huyghens' eye-piece the focal length of the field-lens is three times that of the eye-lens, and the distance between them is twice the focal length of the eye-lens; show that if the lenses be supposed thin, this combination, when focussed for normal vision, will also be in focus (except for aberration) if it be inverted, provided the eye-lens be brought back into the same position as before.

7. Show that if F be the focal length of the object-glass of an astronomical telescope fitted with a Ramsden's eye-piece whose equivalent focal length is f, and d the distance of distinct vision, the magnifying power of the telescope when viewing a very distant object is equal to the ratio

$$F(f+3d) : 3df.$$

8. If F be the focal length of the object-glass of an astronomical telescope, which is fitted with a Ramsden's eye-piece whose field-glass is at a distance a from the object-glass, show that the magnifying power of the telescope is

$$\frac{F}{3(a-F)}.$$

9. Show that in an astronomical telescope fitted with a Ramsden's eye-piece, whatever the distance of distinct vision be, the eye must be placed in front of the eye-lens at a distance $\frac{1}{4}f$ from it, in order to catch all the rays that fall on the field-glass, and that then the magnifying power is equal to $\frac{4}{3}F/f$, F being the focal length of the object-glass and f that of either lens of the eye-piece.

Show that an observer whose distance of distinct vision is less than $\frac{9}{4}f$ cannot make use of the telescope for astronomical measurements.

10. Show that the radius of the stop in an astronomical telescope fitted with a Ramsden's eye-piece, which will intercept all but complete pencils, will be the smaller of the two expressions $\frac{3}{4}r$ and $(4Fr-fR)/(4F+f)$, where F and R are, respectively, the focal length and radius of the object-glass, and f and r similar quantities for either of the two equal lenses which compose the eye-piece.

11. The focal length of the object-glass of an astronomical telescope is 40 inches, and the focal lengths of four convex lenses forming an erecting eye-piece are, respectively, $\frac{3}{2}$, $\frac{1}{6}$, $\frac{3}{8}$, $\frac{3}{2}$ inches, reckoning backwards from the object-glass. The intervals between the first and second and between the second and third being one inch and half-an-inch, respectively, show that when the instrument is in adjustment for eyes which can see with parallel rays, the distance of the eye-lens from the object-glass is $41\frac{1}{2}$ inches, and the magnifying power of the instrument $\frac{88}{3}$.

12. Two telescopes are so arranged that the actual size of the images formed at the foci of their object-glasses can be measured. If a be the ratio of the linear dimensions of the image of any object in the focus of the object-glass of the first telescope to the dimensions of the image of the same object formed in the other, (the telescopes being close together), show that the distance of the object is given by the formula

$$\frac{a-1}{d} = \frac{a}{f} - \frac{1}{f'},$$

f and f' being the focal lengths of the object-glasses.

13. If the focal length of the larger mirror of Newton's telescope be 20 feet, and its diameter 2 feet, find what portion of the incident light is necessarily stopped by the smaller mirror.

14. Show that, if in Gregory's telescope the focal length of the small mirror and of the eye-piece be each 2 inches, and the distance between the foci of the large mirror and of the eye-piece be 32 inches, and the telescope be adjusted so that rays from a distant point emerge in a state of parallelism, then the alteration needed for a person who can see best at a distance of 26 inches will be a motion of the small mirror of approximately ·0005 of an inch.

15. The lenses of a common astronomical telescope whose magnifying power is 16, and length from object-glass to eye-glass $8\frac{1}{2}$ inches, are arranged as a microscope to view an object placed $\frac{5}{8}$ of an inch from the object-glass; find the magnifying power, the least distance of distinct vision being taken to be 8 inches.

16. Show that, if the focussing of the compound microscope were made by adjusting the eye-piece instead of the tube as a whole, the amount of adjustment would be increased in the ratio $(D/f-1)^2$ to 1, approximately, where D is the distance of the focal plane of the eye-piece from the objective, and f is the numerical focal length of the objective.

Show that if the outer surface of the objective, of radius of curvature r, be distant x from an object on which it is focussed in air, the magnification would be diminished $\{1-(\mu-1)\,x/r\}^{-1}$ times, by dipping the face of the objective in liquid of index μ, in which the object has been immersed.

17. Light from an infinitely distant object is reflected obliquely from a slightly convex surface, and the image at the primary focus viewed through an astronomical telescope. The telescope is first focussed for parallel rays, and it is found that to secure distinct vision the eye-piece has to be drawn out a distance x. Show that if F be the focal length of the object-glass, c the distance between the surface and the object-glass, ϕ the angle of incidence, and R the radius of curvature, then

$$R=\frac{2F^2+x(F-c)}{x\cos\phi}.$$

18. Two collimators of the usual construction were pointed directly towards each other and the wires of each were made by adjustment to be seen distinctly together with the images of the wires of the other; the geometrical focal lengths being f and f', ∂f and $\partial f'$ small deviations of the positions of the wires from the geometrical foci, and D the interval between the object-glasses; show that to a first approximation, $-\frac{\partial f}{\partial f'}=\frac{f^2}{f'^2}$, and to a nearer approximation,

$$-\frac{\partial f}{\partial f'}=\frac{f^2}{f'^2}\left\{1+\frac{\partial f}{f}-\frac{\partial f'}{f'}\left(1-\frac{D}{f'}\right)\right\}.$$

19. An object-glass of focal length F is used in combination with a four-glass erecting eye-piece in which the focal lengths of the lenses are f_1, f_2, f_3, f_4 in succession from the object-glass. The distances of the first and last pairs of lenses in the eye-piece are constant and equal to a, b, but the distance x between the middle pair is variable. If the telescope be adjusted for parallelism of the pencils of emergent rays and so that its magnifying power is greatest, the distance y between the focus of the object-glass and the lens f_1 will be given by the equation

$$F\left(\frac{1}{f_1}+\frac{1}{f_2}-\frac{a}{f_1f_2}\right)=2\left(1-\frac{a}{f_1}\right)\left[\left(1-\frac{a}{f_2}\right)-\left(\frac{1}{f_1}+\frac{1}{f_2}-\frac{a}{f_1f_2}\right)y\right]^2,$$

and x will be expressed by a fraction whose numerator and denominator involve y to the first degree only, with known quantities.

CHAPTER XII.

Optical Instruments and Experiments.

272. If light be admitted into a darkened room through an aperture fitted with a single convex lens or a combination of lenses, inverted images of external objects will be formed within the room at their proper distances from the lens; and if the objects be at a considerable distance from the lens, compared with its focal length, the distances of their images will be very nearly the same and equal to that focal length. If therefore a screen be placed perpendicularly to the axis of the lens, at a distance from it equal to the focal length, an inverted picture of the external scene will be formed on the screen. For the purposes of drawing, it is convenient that the image should be thrown into a horizontal position. This is effected by placing between the lens and its principal focus, a plane mirror inclined at an angle of 45^0 to the axis of the lens.

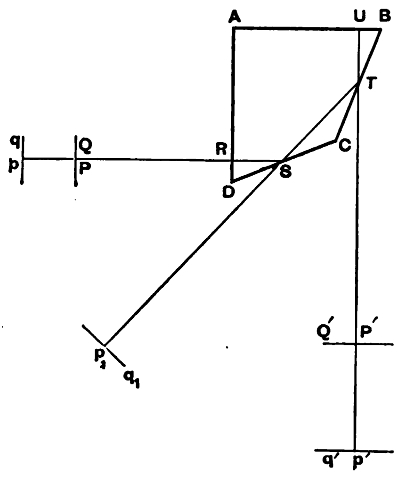
This is the principle of the portable *camera obscura.* A box from which external light is excluded is substituted for the darkened chamber; for the screen may be substituted a sheet of sensitive paper, upon which the light acts chemically; with this arrangement, an inverted picture of the external objects is printed upon the paper, and may be preserved in the form of a *photograph.*

In the *magic lantern* and the *solar microscope*, a picture or an object is placed before a collective lens-system at a distance from it a little greater than the focal length of the system, and is then strongly illuminated by an artificial light or the light of the sun thrown into the axis of the tube by a system of reflectors. A real inverted and magnified image is formed at a certain distance from the lens-system, and may be seen depicted on a screen in a darkened room. If the object and the screen be fixed,

the adjustment may be effected by moving the lens-system backwards or forwards in a sliding tube by means of a screw. The adjustment will always be possible, provided the distance of the screen from the object be greater than four times the focal length of the lens-system.

273. The *camera lucida,* invented by Wollaston, is an instrument of great use to the draftsman, in preparing an accurate drawing of a building or a landscape.

Its essential feature is a quadrilateral prism of glass, represented in the adjoining figure. The angle A is a right angle, and the opposite angle is 135°, while the remaining angles B and D are equal; it follows that the angles B and D are each $67\frac{1}{2}°$. Rays of light which are incident perpendicularly on the face AD and are reflected successively at DC and CB, will emerge perpendicular to the face AB.



Let $PRSTU$ be such a ray, and let PQ be a small object perpendicular to PR. Then an image qp will be formed by refraction at the plane surface AD; the rays diverging from qp will be reflected at the surface CD, and made to proceed from an equal image q_1p_1, symmetrically placed on the other side of CD; the rays diverging from q_1p_1 will be again reflected at the surface CB and another image $q'p'$ will be formed. Finally when the rays proceeding from $q'p'$ are refracted again into the air, they will

proceed from an image $Q'P'$. Let PR the distance of the object from the first surface be denoted by x, and UP' the distance of the final image from the final surface AB by x', and let u, v, w, be the lengths of the three portions of the path within the prism. Then $pR = \mu x$. Also it is easy to see that $Up' = \mu x + u + v + w$, and therefore

$$x' = x + \frac{u+v+w}{\mu}.$$

Hence *the difference between the distances of the object and final image from the vertical and horizontal sides of the prism, respectively, is equal to the length of the path within the prism divided by its refractive index.*

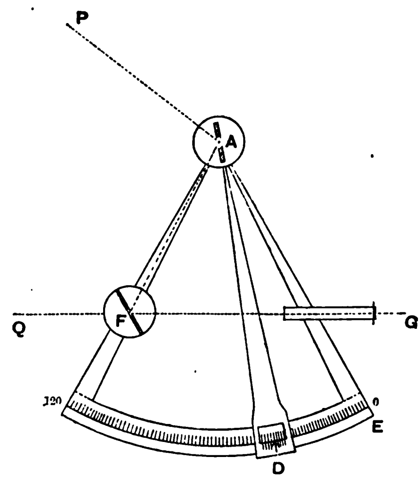
The prism is mounted in a brass frame and attached by its axis to the end of a brass stem, the lower extremity of which may be clamped to a table; the length of the stem may be varied at pleasure by means of a sliding tube. The upper surface of the prism AB is furnished with an eye-stop of small aperture, which is adjusted so that the aperture is as nearly as possible bisected by the edge B; by this means only a small part of the surface AB is used, and the rest is covered. When the vertical face of the prism is turned towards the object, the observer looks downwards through the aperture and sees at the same time the image of the object through the uncovered portion of the prism, and the paper on which it is thrown through the remaining portion of the aperture. The image will be erect, since the rays from the upper part of the object proceed towards the upper part of the image.

Since the dimensions of the prism are very small in comparison with the distance of the object, the distances of the object and image will be nearly equal. If the distance of the object from the prism be very different from the distance of the latter from the table, the image and the paper cannot be seen together distinctly. This may be remedied by a convex lens whose focal length is equal to the greatest distance of the prism from the table. The lens is turned up horizontally under the prism, and the paper being in the principal focus, its image is thrown to an infinite distance and therefore made to coincide with the image of a remote object formed by the prism. The same correction may be made by placing a concave lens of the same focal length vertically in front of

the vertical face of the prism. The rays proceeding from a distant object are made to diverge from an image whose distance is equal to the focal length; this image will therefore coincide with the paper after passing through the prism. The convex lens is to be used by normally sighted persons, the concave by short-sighted persons.

For near objects the adjustment of the distances is completed by varying the distance of the prism from the paper.

274. *Hadley's Sextant* is an instrument for measuring the angular distance between two distant points. It consists of a framework in the form of a sector of a circle, with a graduated



arc, and two plane mirrors, whose planes are perpendicular to the plane of the sector. One of the mirrors A is moveable about an axis through the centre of the arc, and carries a pointer whose vernier slides along the graduated arc. The other mirror is fixed at F and is parallel to the mirror A when the pointer of the latter is at E, the zero of the graduated scale; the lower part of this mirror only is silvered, so that rays of light may be transmitted directly through the upper part. The instrument is fitted with a small telescope G whose axis is directed towards the dividing line of the mirror F.

To measure the angular distance between any two points P, Q, the instrument is brought into the same plane with them and the

telescope G is directed towards one of them Q, which can be seen directly through the unsilvered part of the mirror F. The mirror A is then moved so that P, as seen through the telescope by a pencil reflected in succession at the mirrors A and F, appears to coincide with Q. In this arrangement, the angular distance between the points P and Q is the deviation of the axis of the pencil by the two reflexions; and this is equal to twice the inclination of the mirrors. The inclination of the mirrors may be read off the graduated scale. If the arc be graduated so that every half-degree may be read as a degree, the reading will give the angular distance between the two points without any further calculation.

The Heliostat.

275. A heliostat is an instrument which will reflect the light of the sun in a fixed direction throughout the day, notwithstanding the motion of the sun.

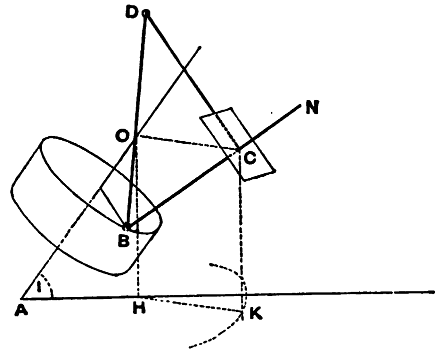
It will be supposed that the change in the Sun's declination during the day is so small that it may be neglected; so that the sun will describe a small circle on the celestial sphere, about the pole.

In all the heliostats which have been constructed, one essential feature is an axis, parallel to the axis of the earth, which is turned by clock-work with the same angular velocity as that of the sun.

The simplest form of heliostat is Fahrenheit's; in this instrument a plane mirror is rigidly connected with the revolving axis, in such a way that the normal to the mirror makes with the axis an angle equal to half the sun's polar distance. If the normal be adjusted so as to have the same right ascension as the sun, they will continue to have the same right ascension throughout the day, and the mirror will continue to reflect the sun's rays in the direction of the earth's axis. By a second fixed mirror they can afterwards be reflected in any required direction.

276. Foucault's heliostat is arranged so as to reflect the solar rays in any required horizontal direction. The point O is fixed, and OA is the rotating axis. OB is a rod rigidly fixed to the revolving axis at an angle which can be adjusted so as to be equal

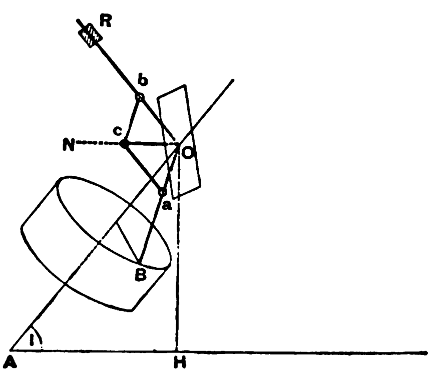
to the sun's polar distance; then if the right ascension of the plane OBA be properly set, OB will represent the direction of the sun's rays throughout the day.



Let OC be the horizontal direction in which the sun's rays are to be reflected, and let C be the point about which a mirror turns; the normal to the mirror NCB is jointed to OB at a point B, such that $OC = OB$. Also the rod BOD passes through a slot CD fixed to the plane of the mirror. Then since $OC = OB$, it follows that OB and OC make equal angles with the normal to the mirror and are always in the same plane with it. Hence sun-light incident on the mirror parallel to OB will be reflected parallel to OC. The line OC can be moved in its own plane to any azimuth.

277. Silbermann has constructed a heliostat which will reflect the sun's rays in any fixed direction.

As before, let OA be the revolving axis, OB a rod rigidly

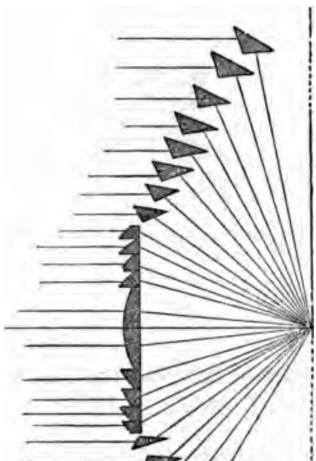


connected with it, so that OB represents the direction of the sun's rays; OR is a rod which can be fixed in any direction, in which it is desired to send the reflected rays. $Oacb$ is a small rhombus of jointed bars, a and b being fixed joints on the rods OB and OR. The normal to the mirror ON carries a slot, in which the angular point c of the rhombus slides. The mirror will then reflect the rays of the sun in the direction OR.

Ex. If a heliostat be arranged so as to reflect the sun's rays in a fixed direction, prove that if the diurnal change in the sun's declination be neglected, the normal to the mirror, and the intersection of the mirror with the plane of reflexion, will describe cones of the second order, whose circular sections are perpendicular to the axis of the earth and to the reflected ray.

Lighthouses.

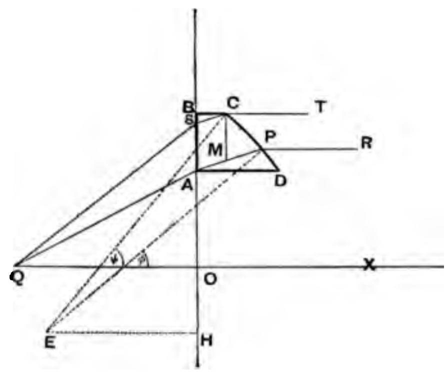
278. The lenses used in lighthouses were first introduced by Fresnel. A lighthouse lens consists of a plano-convex lens surrounded by a series of rings forming steps outwards; it is represented in cross section by the central part of figure. The back



of the compound lens is plane, and the central face is spherical in front. The convex surfaces of the rings are not spherical; they are

annular, generated round the axis of the lens by the revolution of circular arcs in the plane of that axis, but having their centres beyond it in a series of points which retreat further from that axis as each corresponding ring increases in diameter. Fresnel so calculated the coordinates of the respective centres of the actual arcs, so that the extreme rays are made to emerge parallel to the axis. This approximation corrects aberration almost perfectly. In large lighthouses the diameter of this lens subtends an angle of 57° at the centre of the light.

279. To find the form of the cross-section of any ring in Fresnel's lens, let us suppose that Q is the radiant point, $ABCD$ a section of the ring, OAB being the plane side of the compound lens. Let $QAPR$, $QSCT$ be the extreme rays which can pass through the ring in the plane of the figure, the emergent portions PR and CT being parallel to the axis of the lens. Let E be the centre of the circular arc CP, and let the radii EP, EC make angles ϕ, ψ with the axis.



Let α, ρ be the angles of incidence and refraction at A, β and σ those at S, and let $QO=f$, $BC=t$, $AB=b$.

Then $$b=f(\tan\beta-\tan\alpha)+t\tan\sigma,$$

and $$\mu\sin\rho=\sin\alpha,\ \mu\sin\sigma=\sin\beta.$$

Also the angles of incidence and emergence at P are, respectively, $\phi-\rho$, ϕ; those at C are $\psi-\sigma$ and ψ; so that

$$\sin\phi=\mu\sin(\phi-\rho),\ \sin\psi=\mu\sin(\psi-\sigma);$$

and therefore

$$\tan\phi = \frac{\mu\sin\rho}{\mu\cos\rho - 1},$$

$$\tan\psi = \frac{\mu\sin\sigma}{\mu\cos\sigma - 1},$$

from which ϕ and ψ are determined.

Draw CM parallel to AB to meet AP in M.

Then $$CM = b - t\tan\rho.$$

Also from the triangle CPM, the chord CP is easily seen to be given by the equation

$$CP = \frac{CM\cos\rho}{\cos\{\frac{1}{2}(\phi+\psi) - \rho\}}.$$

And therefore if r be the radius of curvature of the arc CP, so that

$$CP = 2r\sin\tfrac{1}{2}(\psi - \phi),$$

we get finally

$$r = \frac{(b - t\tan\rho)\cos\rho}{2\sin\frac{1}{2}(\psi-\phi)\cos\{\frac{1}{2}(\phi+\psi) - \rho\}}.$$

Also, the coordinates of E referred to O as origin are

$$EH = r\cos\psi - t$$

$$OH = r\sin\psi - f\tan\beta - t\tan\sigma,$$

and therefore the curvature of the surface and the position of its centre of curvature are determined.

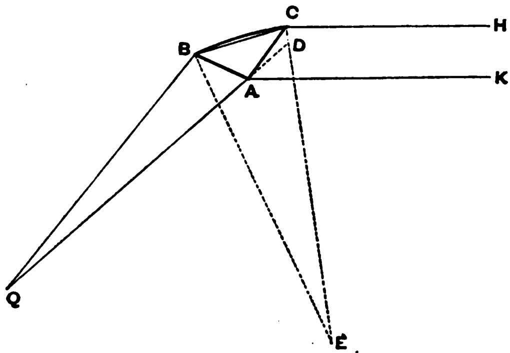
If the section be required for a prism which is detached, $t = 0$.

280. Above and below the lens, there are a series of totally reflecting zones of triangular section. These zones are continued so as to leave only a small space above and below the light, where the light is not caught and sent out horizontally.

We shall now find the form of the section of the reflecting zones, following Fresnel. The reflecting surface is curved; but instead of the true curve a circular arc is necessarily adopted. Let ABC be the section of the totally reflecting zone and Q the radiant point; and let AK, CH be the extreme emerging rays, which are to be parallel to each other and to the horizon. Let the

angle AQB be α, and the angles of incidence and refraction of the ray QA at A be θ, θ'; produce QA to D, and let the angle DAK be δ.

In order to avoid superfluous glass, the side AB of the prism is made to coincide with the path of the ray QB after refraction and



reflexion at B, and the side AC to coincide with the path of the ray QA after refraction at A. Hence the angle BAK is equal to the angle CAQ and therefore the angle BAQ to CAK; also the angle BAC is $\frac{1}{2}\pi + \theta'$; and therefore adding together the several parts which make up the larger angle QAK,

$$2\left(\tfrac{1}{2}\pi - \theta\right) + \tfrac{1}{2}\pi + \theta' = \pi + \delta,$$

or

$$\theta' = 2\theta + \delta - \tfrac{1}{2}\pi,$$

and therefore

$$\sin\theta = \mu \sin\left(2\theta + \delta - \tfrac{1}{2}\pi\right).$$

From this equation the angle θ may be found.

Let ϕ and ϕ' be the angles of incidence and refraction of the ray QB at B, then $\phi = \theta - \alpha$ and therefore

$$\mu \sin\phi' = \sin(\theta - \alpha).$$

The internal incidence of the ray CH at C is equal to θ', since the emergent rays AK, CH are parallel.

At B and C draw the radii BE, CE of the circular arc BC which is the reflecting surface, and join BC; then it is easily seen that the angle

$$ABE = \tfrac{1}{2}\left(\tfrac{1}{2}\pi + \phi'\right), \text{ and } ACE = \tfrac{1}{2}\left(\tfrac{1}{2}\pi + \theta'\right).$$

Also the angle $\quad BEC = BAC - (ABE + ACE)$,

and therefore the angle $BEC = \theta' - \frac{1}{2}(\theta' + \phi')$

$$= \tfrac{1}{2}(\theta' - \phi').$$

The angles EBC and ECB are therefore each equal to

$$\frac{\pi}{2} - \tfrac{1}{4}(\theta' - \phi');$$

therefore the angle $ABC = EBC - ABE = \tfrac{1}{4}(\pi - \theta' - \phi')$

and the angle $ACB = ECB - ACE = \tfrac{1}{4}(\pi - 3\theta' + \phi')$.

Now in the triangle QAB the side QA is supposed known and the angles AQB and QAB; therefore AB is known. Then in the triangle ABC, the two angles B and C are known and also the side AB; therefore the side BC is known.

Lastly the angle BEC has been found, and therefore the radius of curvature of the circular arc is known.

281. So far only the section of the apparatus has been described. This section may be revolved round a vertical axis passing through the light; and this gives the form of a lighthouse apparatus for a fixed light sending light out horizontally in all directions. For a flashing light, it is usual to arrange the lenses in the form of eight panels; each panel consists of a Fresnel's annular lens generated by the revolution of the section about a horizontal axis, together with totally reflecting zones formed by revolving the section through an angle of 45° about the vertical axis. The whole apparatus is then slowly revolved about its vertical axis.

Further information on this subject may be found in a paper by Mr. James T. Chance, "On optical apparatus used in Lighthouses," published in the *Proc. Inst. Civil Engineers*, Vol. XXVI., 1867, and also in the Article "Lighthouses," in the *Encycl. Brit.*

Determination of Refractive Indices.

282. The general method of measuring the refractive index of a solid medium for any particular coloured ray of light, is to observe the minimum deviation of a ray of light of this colour, as it passes through a prism made out of the substance. It has been already seen that, when a ray of light passes through a prism with minimum

deviation, its path is symmetrical with respect to the prism; so that with the usual notation

$$\phi = \psi, \ \phi' = \psi',$$

and therefore

$$\left.\begin{aligned} D + \iota &= 2\phi \\ \iota &= 2\phi' \end{aligned}\right\}.$$

If μ be the refractive index of the medium,

$$\sin \phi = \mu \sin \phi',$$

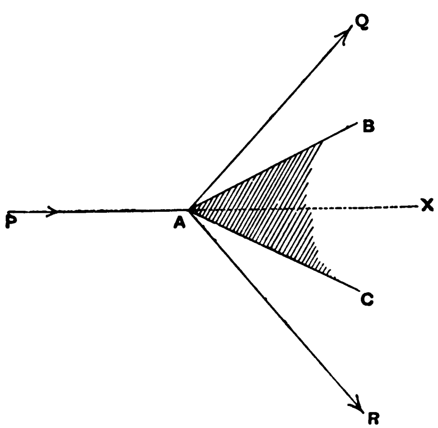
and therefore

$$\sin \tfrac{1}{2}(D + \iota) = \mu \sin \tfrac{1}{2}\iota.$$

As soon, therefore, as we have measured ι, the refracting angle of the prism, and D, the minimum deviation, we can calculate μ.

283. The apparatus used consists essentially of a horizontal graduated circle, with a horizontal telescope which can be turned round, so that its optic axis always passes through the centre of the rim. The prism is fixed with wax or cement to a levelling stand placed over the centre of the graduated circle. The light is supplied through a collimator, which consists of a fine vertical slit placed in the focus of an achromatic object-glass, so that the rays emerge from the collimator parallel to each other, the collimator being fixed so that its axis passes through the centre of the rim.

The refracting angle of the prism is first measured. The prism is placed so that light from the collimator is reflected at both faces of the prism. The image of the slit as reflected at each



face in succession is viewed by means of the telescope, the telescope being moved round till the image falls on the cross-wires of the telescope. The angle through which the telescope must be turned from seeing the image reflected in one face, in order to see the image reflected in the other face, is read off the graduated circle. It may be shown that this angle is equal to twice the refracting angle of the prism. For let BAC be the refracting angle of the prism, and let the incident ray be in direction of PAX. Then, if AQ be the ray reflected in the face AB, AQ and AX must make equal angles with AB, so that

$$\angle BAX = \tfrac{1}{2} \angle QAX.$$

Similarly, if AR be the direction of the ray reflected in the face AC,

$$\angle CAX = \tfrac{1}{2} \angle RAX;$$

and therefore, by addition,

$$\angle BAC = \tfrac{1}{2} \angle QAR.$$

284. The minimum deviation for a ray of definite refrangibility, corresponding to any fixed line of the spectrum, is next measured. The slit is first viewed directly, the prism being turned so as not to obstruct all the light, and the telescope is moved until the line of the spectrum coincides with the cross-wires of the telescope. The prism and telescope are then moved so that an image of the slit formed by light which has passed through the prism is seen through the telescope; and the prism is turned so as to make the image move nearer to the direction of direct light, the telescope following the image so as always to keep it in view. At length a position of the prism is obtained, such that if the prism be turned either way the image recedes from the direction of the direct light; this position of the prism is therefore the position of minimum deviation. The telescope is moved until the line of the spectrum coincides again with the cross-wires of the telescope. The angle through which the telescope has been turned from the position of direct light is read off the graduated circle, and this angle is the minimum deviation required.

285. To measure the refractive index of a liquid, it is enclosed in a hollow prism of glass, made by cementing plates of glass

together. The two sides of the plates however are never accurately parallel, and from the observed deviation it is necessary to subtract the small deviation caused by the empty prism.

The refractive indices of gases in given conditions as to temperature and pressure have been measured by a similar process. They must be enclosed in a tube, the ends of which are closed by two plates of glass placed very obliquely with reference to the axis of the tube.

The experiments of Biot and Arago on the refractive indices of gases showed that for gases the quantity $\mu^2 - 1$ is proportional to the density of the gas, a law which had been enunciated by Newton, who deduced it from his theory of emission.

286. *To find the focal length of a thin convex lens.*

This is usually measured by adjusting the lens and an object, until the distance between the object and the image is a minimum; this distance is then four times the focal length. For, if u, v be the distances of the object and image in front of, and behind, the lens,

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f},$$

while the distance between the object and the image is given by the equation

$$u + v = x.$$

Combining these equations, we get

$$uv = xf,$$

and therefore $$(u - v)^2 = x^2 - 4xf.$$

The quantity $(u - v)^2$ is always positive, and therefore the least value of x is equal to $4f$.

If the lens be concave, it is placed in contact with a convex lens, so that the whole combination may be collective; the focal length of the combination may be determined as before. If f, f' be the numerical focal lengths of the two lenses, F that of the combination,

$$\frac{1}{F} = \frac{1}{f'} - \frac{1}{f},$$

which determines f when f' is known.

On the experimental determination of the focal length and cardinal points of an optical instrument.

287. It has been seen that the position and magnitude of the image of an object formed by any symmetrical optical instrument can always be determined by geometrical construction or by simple formulæ, when the positions of the two focal points and the two principal points are given. The positions of the two principal points are determined as soon as the positions of the focal points and the focal length are known, and thus it appears that the optical system is completely determined when we know the abscissæ of the focal points and the focal length of the system. To determine these three quantities, three experiments will be necessary. Let g, g' be the abscissæ of the two focal points, referred to any origin, ξ, ξ' the abscissæ of the object and its image, respectively; then, if f be the focal length of the system,

$$(g-\xi)(\xi'-g')=f^2.$$

For the sake of symmetry, suppose that all the distances considered are measured from a fixed point whose abscissa is e, and let

$$e-\xi=a,\quad \xi'-e=b,$$
$$e-g=p,\quad g'-e=q;$$

the constants to be determined are then p, q and f.

If we take three different positions of the object and determine by experiment the position of the image for each, we get the three equations

$$(a-p)(b-q)=f^2,$$
$$(a'-p)(b'-q)=f^2,$$
$$(a''-p)(b''-q)=f^2,$$

where a', b' and a'', b'' refer to the second and third positions. The quantities a, b, a', b', a'', b'' are all determined by measuring, and therefore these three equations are sufficient to determine the values of p, q, f in terms of known values. If we eliminate f^2 between the first and second, and again between the first and last, we get two equations of the first degree in p and q, which serve to

determine these quantities without ambiguity. Any one of the three equations will then determine f^2. This will leave the sign of f undetermined; but the sign of f may be found by examining the image. If the image be erect, then $\xi' - g'$ and f will have opposite signs, and if inverted, the same sign.

288. In the formulæ it is supposed that for the three experiments the object lies on one and the same side of the lenses. If in any case the object were at the other side, the positions of the object and image may be interchanged, and then the arrangement is the same as in the preceding cases. Practically we are restricted to real images, and therefore in the case of a single lens we are limited to a collective lens, unless special methods of determining the positions of virtual foci be adopted. It is always possible, however, to combine the system with a single collective lens so as to make the whole system collective, and then, having determined the constants of the combination, to calculate those of the original system.

The experiments should be chosen so as to have the three equations as different as possible, in order that the errors of observation may have the smallest effect on the results.

289. For a single lens, or for an achromatic object-glass, which consists of lenses in contact, the distance between the principal points is usually small. If this distance, which we may call λ, be known, two experiments are all that is necessary. For we then have the equation

$$p + q = 2f + \lambda$$

in addition to the two

$$(a - p)(b - q) = f^2,$$

$$(a' - p)(b' - q) = f^2.$$

Eliminating p and q between these equations, we find

$$\frac{(a' + b' - a - b)^2}{(a' - a)(b - b')} f^2 + 2(a + b + a' + b' - 2\lambda) f - (a + b' - \lambda)(a' + b - \lambda) = 0,$$

which is in general a quadratic to find f. But if the experiments be chosen so that $a' + b' - a - b = 0$, that is, so that the distance of the object from the image is the same in each experiment, while the lens has different positions, the equation reduces to an equa-

tion of the first degree. Let c be the distance between the object and the image; then $a=c-b$, $a'=c-b'$, and the equation becomes

$$4f(c-\lambda)=(c-\lambda+b'-b)(c-\lambda-b'+b),$$

or
$$f=\tfrac{1}{4}(c-\lambda)-\frac{(b'-b)^2}{4(c-\lambda)}.$$

The values of p, q are then

$$p=\tfrac{1}{2}(2f+c+\lambda-b-b'),$$
$$q=\tfrac{1}{2}(2f-c+\lambda+b+b').$$

290. A convenient method of arranging the experiments in this case is first to place the object at a great distance, and then the rays will converge to the second focal point, and thus q is determined immediately. Then reverse the lens, and measure the position of the image of the same object, and this will determine p. Besides these two experiments one other is required, which gives an equation,

$$(a-p)(b-q)=f^2,$$

and determines f. The distance λ is then given by the equation

$$\lambda=p+q-2\sqrt{(a-p)(b-q)}.$$

The solution may now be regarded as complete; but if a specially exact determination is necessary, this investigation may be regarded as giving λ as a preparation to the method just explained.

Gauss gives the following convenient method of arranging the third experiment. On a sheet of paper describe a circle of about the same radius as the lens, and make a small well-defined cross at its centre; then let the lens be placed in contact with the paper so as to be accurately concentric with the circle, and view the cross with a small microscope with cross lines, which is capable of adjustment along the axis of the lens. Adjust the microscope until the image of the cross coincides with the cross in the microscope. Then remove the lens and again adjust the microscope. The distance through which the microscope has been moved is then equal to $\xi'-\xi$. The point of the cross may be taken as the point of reference, and then

$$a=0,\ b=\xi'-\xi.$$

Methods of Photometry.

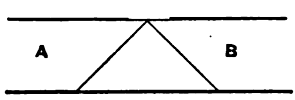
291. It has been shown that when an element of a surface is illuminated by light proceeding from a source of intensity I, at a distance r, so that the axis of the pencil makes an angle θ with the normal to the element of surface, then the intensity of illumination is proportional to

$$\frac{I\cos\theta}{r^2}.$$

It is found that the eye is of itself unable to estimate the ratio of the intensities of two sources of light, but that it is an accurate judge of the equality of illumination of two illuminated surfaces when they are placed side by side. All methods of photometry depend therefore on the equalising of two illuminations.

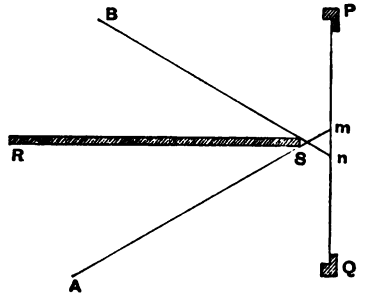
In order to compare the intensities of two sources of light, the two halves of a piece of thin porcelain are illuminated by the two sources, respectively, in such a way that either the light falls normally on the porcelain, or the lights from the two sources make equal angles with the plane of the porcelain. The distances of the lights are then adjusted so that the two halves of the porcelain are equally illuminated. Then the intensities of the sources are in the inverse proportion of the squares of their distances from the porcelain. This is the principle of both Ritchie's and of Foucault's photometers.

292. Ritchie's photometer consists of a rectangular box open at both ends. In the lid is a narrow strip of porcelain or oiled paper. The instrument is placed between the two sources to be compared, and the light is reflected up to the porcelain by two pieces of mirror (which must be cut from the same piece of glass)



placed at angles of 45° to the axis of the box. The box is then moved from one source towards the other until the two halves of the porcelain are equally illuminated, and the distances of the lights measured.

293. In Foucault's photometer the lights which are to be compared act separately on two different parts of the same vertical plate of thin transparent porcelain, PQ. RS is an opaque vertical screen which separates the two illuminations from one another. If this screen be so adjusted that the vertical planes ASm, BSn which limit the regions illuminated separately by the two sources A, B, intersect just in front of the lamina PQ, the dark band mn can be bright



made as narrow as we please. The distances of A and B are then adjusted so that the two portions of the lamina are equally illuminated.

294. In Rumford's photometer the intensities of the two shadows on a screen of a vertical rod due to the two lights are compared. The lights are arranged so that the shadows fall close together, and the shadow formed by one light is lighted by the light from the other source. The distances being so adjusted that the shadows are of equal intensity, the distances of the lights are measured, and thus the intensities of the two sources can be compared.

Bunsen has invented a very simple photometer. If a spot of grease be made on a sheet of paper, then if the paper be equally illuminated on its two sides, the transparent spot cannot be seen except by close inspection. The sources of light are placed on opposite sides of the paper and their distances are so adjusted that the grease spot disappears; then the intensities of the sources are inversely as the squares of their distances from the paper. The adjustment should first be made from the side on which one source lies, then the screen should be turned round and the adjustment made from the side on which lies the other source, the same side

of the paper being observed each time. The mean of these two positions will give a fairly accurate result.

295. In all these comparisons the lights are supposed to be of the same quality, otherwise the comparison fails. A strict comparison of two compound lights of different qualities could only be arrived at after comparing the relative intensities of all the different coloured rays of the spectra given by the two lights, and tabulating the results.

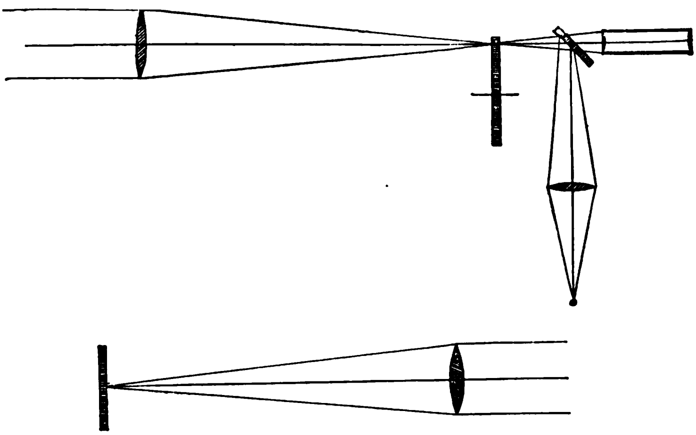
296. The first steps towards stellar photometry were taken by Sir John Herschel. He received the light of the moon on a lens of short focus, so as to make a small image of the moon in the focus of the lens; this image he used as an artificial star, with reference to which the brightness of stars could be estimated. The lens could be adjusted at different distances until the brightness of the star and the image were equal. The distances of the image for different stars give a means of comparing their intensities.

Dr Seidel used an instrument not very different in principle but more convenient in practice. He divided the small object-glass of a telescope into two halves, one of which could be moved in the direction of its axis. Two stars to be compared were made to appear nearly in the same direction by internal prismatic reflexion. The distance through which the half of the object-glass had to be moved in order that the images might appear of equal intensity gave sufficient data for a comparison of the brightness of the stars.

297. More recently a method of comparing the brightness of stars, depending on the fact that the absorption of light passing through a dense medium is a function of the thickness of the medium, has been used by Professor Pritchard at the Observatory at Oxford. A thin wedge of homogeneous and nearly neutral-tinted glass is interposed, so that the star images formed in the focus of the telescope are seen through the wedge. Simple means are contrived for measuring with great exactness the several thicknesses at which the light of these telescopic star images is extinguished. In this way the light of any star may be readily compared with that of any standard star, and a catalogue of star-magnitudes can be formed.

Methods of determining the Velocity of Light.

298. There are two methods of determining the velocity of light by optical experiments, the one devised by Fizeau and the other by Foucault. Fizeau's experiments were repeated in 1876 by M. Cornu, and later a modification of Fizeau's method has been used by Dr. Young and Professor Forbes in Scotland. The velocity of light has also been determined by A. A. Michelson, of the United States navy, who followed Foucault's method.



299. In Fizeau's experiments two astronomical telescopes several miles apart are arranged so that their axes are accurately parallel, the one telescope looking into the other. In one of the telescopes a mirror is then placed at the focus of the object-glass, exactly perpendicular to the axis of the instrument. The observer stands at the other telescope; in this instrument a plate of glass, inclined at an angle of 45° to the axis of the telescope, is placed between the eye-piece and the principal focus of the object-glass. Light is admitted through the side of the instrument and reflected down the tube by the plate glass, the rays coming to a focus at the principal focus of the object-glass, so that they may emerge from the instrument in a direction parallel to its axis. These rays of light enter the object-glass of the distant telescope, are reflected back in the same direction by its mirror, and some of these rays after passing the object-glass will pass through the inclined plate of

glass and enter the eye-piece and will be received by the eye in the usual manner. A wheel with a large number of fine teeth is rotated, so that the teeth pass in front of the focus of the object-glass. When the wheel rotates comparatively slowly, but quickly enough for the intermittent light to make a continuous impression on the eye, the eye will see an image of the light; for the time taken to travel to the distant telescope and back again is so small that light which passed through the space between two teeth at starting will have time to return through the same space before the wheel has turned appreciably. We shall suppose that the breadth of the teeth is equal to the interval between two consecutive teeth. If now the speed of rotation be increased, it may happen that light which passed through the space between two teeth, may on its return be stopped by the next tooth, which has moved forwards in the interval. In this case no light will reach the eye. If the velocity of rotation be continuously increased, the image will reappear, at first faintly, then more brightly, and will again begin to disappear, and so on. Let $2l$ be the whole length of the path of the light as it passes from the toothed wheel back to the same point. Then if v be the velocity of light, the time taken for the complete journey to and fro will be $2l/v$. Let m be the number of teeth in the wheel, and n the number of revolutions of the wheel per second; then the time taken by one tooth to pass before the principal focus will be $1/2mn$ seconds. If therefore the number of revolutions per second be such as to produce the first eclipse,

$$\frac{2l}{v} = \frac{1}{2mn},$$

or

$$v = 4mnl;$$

and if n be such as to cause the p^{th} eclipse, it may easily be seen that

$$v = \frac{4mnl}{2p-1}.$$

The distance l and the number of revolutions per second are observed, and then v is determined by these formulæ.

The imperfection in this method is that in actual experiments a total eclipse of the reflected rays is hardly ever reached; there is usually only a very great falling off in its intensity, and the

exact moment which must be taken to represent the moment of eclipse cannot be determined with very great precision.

300. Messrs. Young and Forbes used a telescope arranged with a rotating wheel, similar to the instrument described, with two distant reflecting telescopes nearly in the same line, but at different distances. The method of observation was to arrange the speed of the toothed wheel so that the brightness of the two images seen should be equal; it was found that this could be effected with considerable precision.

Let E be the brightness of the image when the wheel is not in position; then when the wheel is rotating slowly the brightness will be $\frac{1}{2}E$.

As before, let n be the number of revolutions per second, and let t be the time occupied by the double journey; also let k be the breadth of each tooth and interval in the wheel. In the time t, the circumference of the wheel will have passed over a space $2mnkt$. Before the first eclipse, it is easy to see that the effect of the rotation of the wheel is the same as if the breadth of each tooth were $k+2mnkt$, while the wheel revolved slowly. Thus, if I be the intensity of the light,

$$I = \tfrac{1}{2}E\{1 - 2mnt\}.$$

If N denote the number of revolutions per second at the first eclipse,

$$1 - 2mtN = 0,$$

or
$$N = \frac{1}{2mt} = \frac{v}{4ml}.$$

Thus in the first phase

$$I = \tfrac{1}{2}E\left\{1 - \frac{n}{N}\right\}.$$

In the second phase, that is, when n passes the value N, I is increasing, and is represented by the formula

$$I = \tfrac{1}{2}E\left\{\frac{n}{N} - 1\right\}.$$

In the same way it may be seen that in the p^{th} phase, when p is odd, we have

$$I = \tfrac{1}{2}E\left\{p - \frac{n}{N}\right\},$$

and when p is even

$$I = \tfrac{1}{2}E\left\{\frac{n}{N} - p + 1\right\}.$$

Let E', l', t', N', I' denote quantities for the second distant telescope, similar to those denoted by the same letters for the other. We shall denote the distant telescopes by A and B, and shall suppose that A is the more distant. As before, it may be proved that in the p^{th} phase of B

$$I' = \tfrac{1}{2}E'\left\{p - \frac{n}{N'}\right\},$$

or

$$I' = \tfrac{1}{2}E'\left\{\frac{n}{N'} - p + 1\right\},$$

according as p is odd or even.

By comparing the values of I and I', it may be seen that the r^{th} equality may be in the r^{th} phase of B and the $(r+1)^{\text{th}}$ phase of A.

If r be even, the condition for the r^{th} equality will therefore be

$$\tfrac{1}{2}E\left\{r + 1 - \frac{n}{N}\right\} = \tfrac{1}{2}E'\left\{\frac{n}{N'} - r + 1\right\},$$

and the condition for the $(r+1)^{\text{th}}$ equality will be

$$\tfrac{1}{2}E\left\{\frac{n'}{N} - r - 1\right\} = \tfrac{1}{2}E'\left\{r + 1 - \frac{n'}{N'}\right\}.$$

If we subtract these equations, we get

$$E\left\{\frac{n+n'}{N} - (2r+2)\right\} = E'\left\{2r - \frac{n+n'}{N'}\right\},$$

or

$$\tfrac{1}{2}(n+n')\left\{\frac{E}{N} + \frac{E'}{N'}\right\} = (r+1)E + rE'.$$

A very great simplification may be effected in this formula by choosing the two distances l and l' so that

$$\frac{l}{l'} = \frac{r+1}{r},$$

where r is an even integer. In the actual experiments of Messrs. Young and Forbes, $l : l' = 13 : 12$.

Then $N'/N = (r+1)/r$, and therefore the equation of condition becomes

$$\tfrac{1}{2}(n+n')\frac{1}{N'r} = 1,$$

or

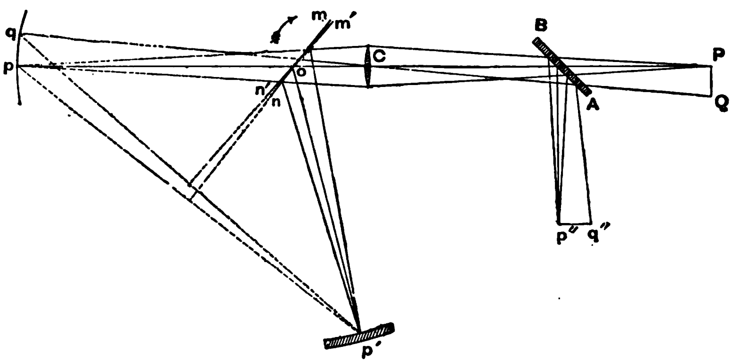
$$N' = \frac{n+n'}{2r}.$$

But it has been shown that $N' = v/4ml'$; and therefore we get,

$$v = \frac{2ml'(n+n')}{r}.$$

The value obtained by these experiments is 301,382,000 metres per second. The value found by Cornu, using Fizeau's method was 300,400,000 metres per second.

301. We shall next give a short account of Foucault's experiments to determine the velocity of light.



A beam of sunlight was transmitted by means of a mirror into a dark room through a small square hole in the window-shutter, and after passing through a lens C was allowed to fall on a small plane mirror *mon* which was capable of rapid rotation about an axis through o perpendicular to the plane of the paper. At present we shall confine ourselves to the consideration of the path of a small pencil of the incident light which diverges from a point P of the aperture. This pencil, after passing through the lens, is made to converge to the point p; but before the rays reach p they are intercepted by the plane mirror *mon* and are reflected to the point p', when $op = op'$. At p' is placed a portion of a spherical mirror whose centre is o and radius op, which reflects the pencil back

again in the same direction, and if the small plane mirror be at rest the pencil will retrace its original course back to P. Between P and the lens C is placed a sheet of plate glass, inclined at an angle of 45° to the axis PC; and part of the returning pencil is reflected at this piece of glass and is brought to a focus at p'', where it is viewed through a telescope. When the mirror *mon* is made to revolve slowly, the light will be returned only when the mirror *mon* is in a position to send light to the small mirror at p', and therefore the image p'' will be intermittent; but if the velocity of rotation be increased up to about 30 revolutions per second, the impression produced in an observer's eye is continuous. So long as the mirror revolves with moderate velocity, the time taken by the light to travel from o to p' and back again is so short that the returning pencil reaches the mirror *mon* before it has appreciably changed its position; but if the velocity of rotation be greatly increased, until the mirror makes several hundred rotations per second, the mirror will have turned through a small angle during the time occupied by the reflected light in passing from o to p' and back again. The pencil returning from p' will be reflected by the mirror in its new position, and after reflexion will appear to diverge from a point q, where $oq = op'$, and after passing through the lens will be made to converge to a point Q on the line qC; the image by reflexion in the plate glass will therefore be at q'' instead of p'', where $p''q'' = PQ$.

Across the aperture through which the light was admitted was stretched a fine wire, whose position is represented by P, and the displacement of the image of this wire $p''q''$ can be measured by the aid of the observing telescope. Let the value of this displacement be δ.

302. Let n be the number of revolutions of the mirror per second; this can be determined by means of a siren. Also let $CP = a$, $Co = b$, and let $op' = r$. Then if v be the velocity of light, the time occupied by the light in passing from o to p' and back again will be

$$t = \frac{2r}{v}.$$

During this time the revolving mirror will have rotated through an angle $2\pi nt$ or $4\pi nr/v$.

The points p, q, p' lie on the circle whose centre is o; also the lines pp', qp' are respectively perpendicular to the two positions of the mirror, and therefore the angle $pp'q$ is equal to the angle between the two positions of the mirror, or to $4\pi nr/v$. It therefore follows that the arc pq subtends at the centre of the circle an angle $8\pi nr/v$; and therefore

$$pq = \frac{8\pi nr^2}{v}.$$

Also, by similar triangles, $PQ : pq = a : (b + r)$,

and therefore $$PQ = \frac{8\pi nr^2 a}{v\,(b+r)}.$$

This length PQ, being equal to $p''q''$, has been determined by observation to be δ, and therefore we get

$$v = \frac{8\pi nr^2 a}{\delta\,(b+r)},$$

an equation which expresses v in terms of quantities which can be measured.

Foucault found the velocity of light by this method to be 298,000,000 metres per second. The value obtained by Michelson by a slight modification of the same method was 299,940,000 metres per second.

The method employed by Foucault may be applied to the determination of the velocity of light in other transparent media, such as water. For this purpose a tube filled with the water, with its ends closed by plate glass, is placed between the revolving mirror and the small spherical mirror, so that part of the double journey is performed through water instead of air. It is found that light travels slower in water than in air.

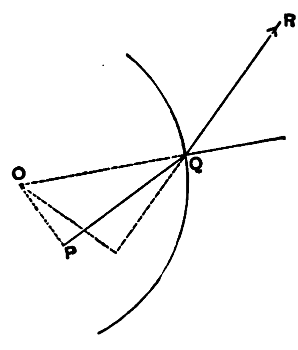
CHAPTER XIII.

Refraction through Media of Varying Density. Meteorological Optics.

303. When the medium varies continuously according to a given law we may regard the refractive index at any point as a given function of the coordinates of that point. Equating this function to a constant, we obtain the equation to a surface along which the refractive index is constant; the form of this surface will indicate the manner of stratification of the medium. By considering the refraction of a ray of light as it passes from one stratum of uniform refractive index to a consecutive stratum, we are led to a differential equation to the path of the ray; the solution of this equation will determine the equation to the path.

304. We shall first suppose the medium symmetrical about a point, that is, stratified in concentric spherical surfaces, the ray moving in a plane passing through this point.

Let PQ, QR be two consecutive directions of the ray, after



being refracted at a spherical surface whose centre is O. Then if ϕ, ϕ' be the angles of incidence and refraction,

$$\mu \sin \phi = \mu' \sin \phi'.$$

If p, p' be the lengths of the perpendiculars drawn from O to the ray before and after refraction, this equation may be expressed in the form,

$$\mu p = \mu' p'.$$

This result is true for any such refraction; and therefore if the ray passes through a continuously changing medium, stratified in spherical surfaces whose centre is O, the equation of the ray may be expressed by the equation

$$\mu p = a,$$

where a is a constant.

We now can easily find the law of the refractive index in the medium so that a given curve may be described. From the equation of the curve we can express p in terms of r, and then the law of variation of refractive power is given by the equation

$$\mu = \frac{a}{p}.$$

Ex. 1. If μ varies inversely as the radius vector, show that the path of any ray is an equiangular spiral.

Ex. 2. If μ varies inversely as r^{n+1}, the equation to the path of a ray is

$$r^n = a^n \cos n\theta.$$

Ex. 3. If $\mu \propto \frac{1}{\sqrt{r^2 - a^2}}$, the path of a ray is an epicycloid.

305. As an illustration of the way in which objects are seen in a heterogeneous medium, let us consider a medium such that

$$\mu = \frac{b}{a^2 + r^2},$$

where a and b are fixed constants. This law of refractive index was suggested to Maxwell by the eye of a fish. The equation to the path of any ray is $\mu p = \text{const.}$, or

$$\frac{bp}{a^2 + r^2} = \text{constant} = \frac{b}{2c}, \text{ say},$$

where c is an arbitrary constant which varies as we pass from one ray to another. The equation to any ray is, therefore,

$$2cp = a^2 + r^2,$$

from which we derive the equation

$$\frac{rdr}{dp} = c.$$

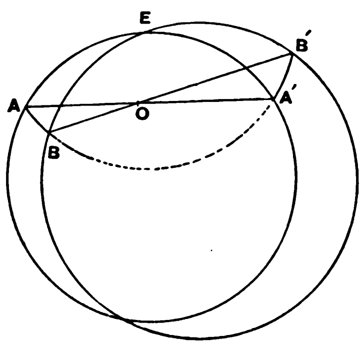
In words, this result shows that the radius of curvature of the curve is the same at all points along it, so that the path of the ray is a circle whose radius is c. Indeed it is easy to see that the relation between r and p corresponding to any point of a circle whose radius is c, and whose centre is at a distance k from the origin is

$$k^2 = r^2 + c^2 - 2cp,$$

so that $$a^2 = c^2 - k^2.$$

Thus if AOA' be a chord of the circle, through the origin, the rectangle $AO \,.\, OA' = a^2$. This result is independent of the particular ray chosen, and therefore A, A' are conjugate points. Pairs of conjugate points are therefore situated on the same line through the centre of the spherical strata, and the product of their distances from that centre is equal to a^2.

Now suppose that an eye is viewing an object through a medium of this kind. We shall suppose that the eye is placed in a small crevasse bounded by orthotomic surfaces, and that the eye is in air close to the surface of the crevasse. Let AB be a small object, $A'B'$ its image, and let E be the position of the eye. Then when the eye is directed towards the object it will see it erect. But



when the eye is turned away from the object it will still see it in the position $A'B'$, inverted. Moreover if we trace the rays by which the eye sees the latter image, it is clear that they come from the *back* of the object, so that it is the *back* of the object which is seen inverted at $A'B'$.

There is another peculiarity about the images thus formed. The amount of divergence in the plane of the figure will not in general be the same as that perpendicular to its plane; the rays will therefore have a different divergence for height and breadth.

306. *Astronomical refraction* is the name given to the angle between the apparent direction of a star as seen through the atmosphere, and the direction in which it would appear if there were no atmosphere.

Without sensible error the earth may be considered spherical, and the atmosphere stratified in spherical layers whose centre is the centre of the earth. It has already been proved that the path of a ray of light through such an atmosphere will be such that

$$\mu p = \text{const.},$$

where p is the perpendicular from the centre of the earth on the tangent to the path at a point where the refractive index is μ. Let x be the radius vector drawn from the centre to any point of the path; this radius vector will be the normal to the stratum. Also let ϕ be the angle between the ray and this normal, then the preceding equation may be written in the form

$$\mu x \sin \phi = \mu_0 a \sin z,$$

where μ_0, a, z are the values which μ, x, ϕ assume at the surface of the earth.

Also, if the consecutive element of the path of the ray make an angle ϕ' with the normal, the deviation $\phi - \phi'$ will be the increment of the atmospherical refraction. If therefore we denote the atmospherical refraction by r, $\phi - \phi' = \partial r$. The law of refraction is

$$\mu \sin \phi = (\mu + \partial\mu) \sin (\phi - \partial r),$$

or

$$\mu \sin \phi = (\mu + \partial\mu)(\sin \phi - \partial r \cos \phi);$$

that is

$$\partial\mu \sin \phi - \mu \partial r \cos \phi = 0,$$

which gives

$$\frac{dr}{d\mu} = \frac{\tan \phi}{\mu}.$$

If we substitute the value of ϕ in terms of z, as obtained by the previous equation, we find

$$\frac{dr}{d\mu} = \frac{1}{\mu}\frac{\mu_0 a \sin z}{\sqrt{\mu^2 x^2 - \mu_0^2 a^2 \sin^2 z}},$$

or

$$r = \mu_0 a \int \frac{\sin z \, d\mu}{\mu\sqrt{\mu^2 x^2 - \mu_0^2 a^2 \sin^2 z}}.$$

Before this integral can be evaluated, it is necessary to know the relation between μ and z.

307. Simpson assumed that the law of decrease of density of the atmosphere was such that some power of the refractive index varies inversely as the distance from the centre of the earth; this hypothesis is represented by the equation

$$\left(\frac{\mu}{\mu_0}\right)^{n+1} = \frac{a}{x}.$$

From this equation we deduce

$$\sin \phi = \left(\frac{\mu}{\mu_0}\right)^n \sin z.$$

If we take logarithms, and differentiate this equation, it becomes

$$\frac{d\phi}{\tan \phi} = \frac{n d\mu}{\mu},$$

so that

$$dr = \frac{d\phi}{n}.$$

To determine the limits of integration of this equation, it may be supposed that at a great distance from the earth's surface the air becomes so rarefied that its action on the path of light may be neglected. If θ be the value of ϕ for this straight path, we get

$$\sin \theta = \left(\frac{1}{\mu_0}\right)^n \sin z.$$

Also by integration

$$r = \int_\theta^z \frac{d\phi}{n} = \frac{1}{n}(z - \theta);$$

and therefore finally, substituting for θ, we get the value of the astronomical refraction,

$$r = \frac{1}{n}\left[z - \sin^{-1}\left(\frac{\sin z}{\mu_0^n}\right)\right].$$

This is Simpson's formula of refraction.

308. Bradley expressed this formula in another form. Simpson's formula may be written

$$\frac{\sin z}{\sin (z - nr)} = \mu_0^n,$$

and therefore $$\frac{\sin z - \sin (z - nr)}{\sin z + \sin (z - nr)} = \frac{\mu_0^n - 1}{\mu_0^n + 1};$$

whence $$\tan \frac{nr}{2} = \frac{\mu_0^n - 1}{\mu_0^n + 1} \tan \left(z - \frac{nr}{2}\right),$$

or approximately, $$r = \frac{2}{n} \frac{\mu_0^n - 1}{\mu_0^n + 1} \tan \left(z - \frac{nr}{2}\right).$$

Bradley wrote this formula in the form

$$r = g \tan (z - fr)$$

and found that for a mean state of the air, corresponding to the barometer 29·6 inches, and thermometer 50° Fahrenheit, we can express the observed refractions very closely, by taking

$$g = 57''{\cdot}036, f = 3.$$

309. It has been proved by experiments of Biot and Arago, that if μ be the refractive index and ρ the density of the atmosphere,

$$\mu^2 - 1 = 4k\rho,$$

where $4k$ is a constant determined by experiments. It is found to be given by the equation

$$4k = {\cdot}000588768.$$

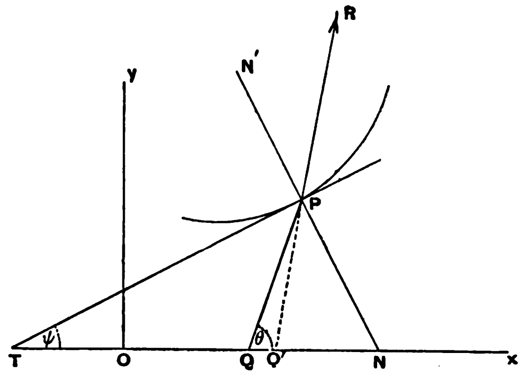
If the density ρ be expressed in terms of x, by the theory of equilibrium of the atmosphere, the accurate relation between μ and x may be determined. This has been done by Laplace, and more completely by Bessel, but the investigation is intricate and scarcely belongs to the province of Optics. For a full account of Bessel's investigation we may refer to Chauvenet's *Astronomy*, Vol. I.

310. Next suppose the ray to move in a plane, through a variable medium which is symmetrical with regard to the plane; to find the intrinsic equation of the path.

We shall suppose that the refractive index is defined by a known law,

$$\mu = f(x, y).$$

Let P be the point of incidence of the ray on any stratum of constant refractive power. Let QP be the direction of ray before refraction, $Q'PR$ the direction after refraction, and let these



directions make angles θ, $\theta + \partial\theta$, respectively, with the axis of x. Draw the tangent PT, and the normal NPN', to the curve of uniform refractive index which passes through P, and let PT make an angle ψ with the axis. Then by the law of refraction

$$\mu \sin QPN = (\mu + \partial\mu) \sin RPN' = (\mu + \partial\mu) \sin Q'PN;$$

that is,

$$\mu \cos(\theta - \psi) = (\mu + \partial\mu) \cos(\theta + \partial\theta - \psi).$$

Retaining only first powers of the small quantities $\partial\mu$, $\partial\theta$, this equation reduces to

$$\partial\mu \cos(\theta - \psi) = \mu\, \partial\theta \sin(\theta - \psi).$$

Also, since PT is the tangent to the curve whose equation is μ = constant,

$$\frac{d\mu}{dx} \cos\psi + \frac{d\mu}{dy} \sin\psi = 0;$$

also $$\cos\theta = \frac{dx}{ds},\ \sin\theta = \frac{dy}{ds};$$

and from these three equations we deduce

$$\tan(\theta-\psi) \equiv \frac{\tan\theta - \tan\psi}{1+\tan\theta\tan\psi},$$

$$= \frac{\dfrac{d\mu}{dy}\dfrac{dy}{ds} + \dfrac{d\mu}{dx}\dfrac{dx}{ds}}{\dfrac{d\mu}{dy}\dfrac{dx}{ds} - \dfrac{d\mu}{dx}\dfrac{dy}{ds}};$$

and this may be written in the form,

$$\frac{d\mu}{ds}\cot(\theta-\psi) = \frac{d\mu}{dy}\frac{dx}{ds} - \frac{d\mu}{dx}\frac{dy}{ds}.$$

But the equation of refraction may be written

$$\frac{d\mu}{ds}\cot(\theta-\psi) = \mu\frac{d\theta}{ds} = \frac{\mu}{\rho},$$

where ρ is the radius of curvature of the path of the ray; and therefore the intrinsic equation of the path of the ray is

$$\frac{\mu}{\rho} = \frac{d\mu}{dy}\frac{dx}{ds} - \frac{d\mu}{dx}\frac{dy}{ds}.$$

Since $-\dfrac{dy}{ds}$ and $\dfrac{dx}{ds}$ are the direction cosines of the normal, measured in the direction of ρ, this equation may be written

$$\frac{\mu}{\rho} = \frac{d\mu}{dn},$$

where dn is an element of the normal to the curved ray; or finally

$$\frac{1}{\rho} = \frac{d}{dn}(\log\mu).$$

311. If the medium be stratified in horizontal layers, the refractive index is a function of y only, and the angle ψ is zero. The previous investigation then gives

$$\partial\mu\cos\theta = \mu\,\partial\theta\sin\theta,$$

or

$$\partial(\mu\cos\theta) = 0.$$

By integration, we get

$$\mu\cos\theta = c,$$

where c is a constant; an equation which might have been deduced directly from the law of refraction. The differential equation of the path is therefore

$$\mu^2 = c^2 \left\{1 + \left(\frac{dy}{dx}\right)^2\right\},$$

or

$$\frac{dx}{c} = \pm \frac{dy}{\sqrt{\mu^2 - c^2}}.$$

As soon as the form of μ is given in terms of y, this equation can be integrated, and the equation of the path determined.

The form of the curve is symmetrical about an axis parallel to that of y. To find the position of the vertex, we have only to make the tangent horizontal, or

$$\frac{dy}{dx} = 0;$$

we then find

$$\mu^2 = c^2.$$

If we make the ray pass through a point $(0, b)$, as for instance through the eye, we find a locus of vertices. Writing $\phi(y)$ for μ^2, the equation to a ray passing through the point $(0, b)$ is

$$x = c\int_b^y \frac{dy}{\sqrt{\phi(y) - c^2}}.$$

But the vertex of this curve is found by combining its equation with the equation

$$\phi(y) = c^2,$$

and therefore if (ξ, η) be the vertex of any ray passing into the eye,

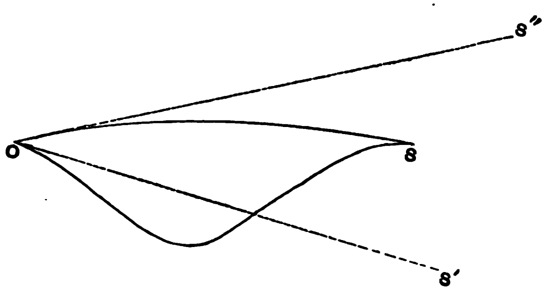
$$\xi = \sqrt{\phi(\eta)}\int_b^\eta \frac{dy}{\sqrt{\phi(y) - \phi(\eta)}}.$$

To find where an object close to the horizon would be seen, the eye being on the same level as the object, we must trace the curve of vertices of all the rays passing into the eye, and find the points where it is met by a vertical half-way between the eye and the object; each of the points of intersection is a vertex of the path of a ray by which the object can be seen. When the curve of vertices at one of these points leans forwards towards the eye, two

contiguous rays cross each other and an inverted image is seen; but if the curve of vertices leans away from the eye, the contiguous rays do not cross each other and the image is seen erect.

312. Usually the density of the air decreases with the height above the ground; but often in countries where there are large tracts of hot sand, the air is heated and rare close to the ground, and for a small distance the density increases as we rise from the ground, but afterwards diminishes. At the height where the density is a maximum, we shall have μ stationary, so that $\frac{d\mu}{dy}=0$; and from the equation to the ray we infer that $\frac{d^2x}{dy^2}$ is zero, so that the path of the ray has a point of inflexion.

If S be an object and O the observer's eye, both situated above



the layer of maximum density, a ray passing from S to O by the upper air will be concave to the horizon.

If we consider rays proceeding from S at a less inclination to the horizon, some of them will remain concave, but those more inclined to the horizon may have a point of inflexion, and in this case, if the ray be not stopped by the ground, it may reach the eye by another path. Thus the observer will see the object directly and erect by the upper path, in the direction OS'', and an inverted image in the direction OS' due to the lower path. The appearance will be the same as if an upper erect object at S'' were reflected in a mirror or lake.

At sea this phenomenon is often seen turned upside down. The density of the air decreases rapidly from the surface of the water upwards. An image of a distant ship or shore is

thus often seen erect through the nearly uniform lower strata of the air, while just above them is seen an inverted image, formed by rays which travel along paths passing through the upper strata. These phenomena are known as *mirage*, and the explanation was first given by Monge.

313. Let the refractive index be defined according to any continuous law,

$$\mu = f(x, y, z).$$

The path of any ray will be such as to make $\int \mu ds$ a minimum. Let $V = \int \mu ds$, taken between any two points A, B. Then, if the path be varied slightly,

$$\partial V = \int \partial \mu \, ds + \int \mu \, \partial (ds).$$

Also, since $ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$,

$$\partial (ds) = \frac{dx}{ds} \partial (dx) + \frac{dy}{ds} \partial (dy) + \frac{dz}{ds} \partial (dz),$$

and therefore the second integral may be written

$$\int \left\{ \mu \frac{dx}{ds} \partial (dx) + \mu \frac{dy}{ds} \partial (dy) + \mu \frac{dz}{ds} \partial (dz) \right\}.$$

But $\partial (dx) = d (\partial x)$; hence, integrating by parts, this becomes

$$\left[\mu \frac{dx}{ds} \partial x + \mu \frac{dy}{ds} \partial y + \mu \frac{dz}{ds} \partial z \right]_A^B$$

$$- \int \left\{ \frac{d}{ds} \left(\mu \frac{dx}{ds} \right) \partial x + \frac{d}{ds} \left(\mu \frac{dy}{ds} \right) \partial y + \frac{d}{ds} \left(\mu \frac{dz}{ds} \right) \partial z \right\} ds.$$

Collecting the terms, the value of ∂V becomes

$$\partial V = \left[\mu \left(\frac{dx}{ds} \partial x + \frac{dy}{ds} \partial y + \frac{dz}{ds} \partial z \right) \right]_A^B$$

$$+ \int \left[\left\{ \frac{d\mu}{dx} - \frac{d}{ds} \left(\mu \frac{dx}{ds} \right) \right\} \partial x + \left\{ \frac{d\mu}{dy} - \frac{d}{ds} \left(\mu \frac{dy}{ds} \right) \right\} \partial y \right.$$

$$\left. + \left\{ \frac{d\mu}{dz} - \frac{d}{ds} \left(\mu \frac{dz}{ds} \right) \right\} \partial z \right] ds.$$

By the principles of the Calculus of Variations, ∂V must vanish

for all indefinitely small variations of the path; so that all along the ray we get the relations

$$\left.\begin{aligned}\frac{d}{ds}\left(\mu\frac{dx}{ds}\right)&=\frac{d\mu}{dx}\\ \frac{d}{ds}\left(\mu\frac{dy}{ds}\right)&=\frac{d\mu}{dy}\\ \frac{d}{ds}\left(\mu\frac{dz}{dx}\right)&=\frac{d\mu}{dz}\end{aligned}\right\}.$$

The terminal condition is

$$\frac{dx}{ds}\partial x+\frac{dy}{ds}\partial y+\frac{dz}{ds}\partial z=0$$

at each end. If we suppose that the terminal points A, B are constrained to move on surfaces for which V is constant, this equation expresses the fact that at each end, the ray is perpendicular to the surfaces $V=$ constant.

Any two of the above general equations, when integrated, will give the path of the ray.

The direction cosines of the radius of curvature of the path are $\rho\frac{d^2x}{ds^2}$, $\rho\frac{d^2y}{ds^2}$, $\rho\frac{d^2z}{ds^2}$. If dn be an element of this principal normal,

$$\begin{aligned}\frac{d\mu}{dn}&=\rho\left\{\frac{d^2x}{ds^2}\frac{d\mu}{dx}+\frac{d^2y}{ds^2}\frac{d\mu}{dy}+\frac{d^2z}{ds^2}\frac{d\mu}{dz}\right\}\\ &=\mu\rho\left\{\left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2+\left(\frac{d^2z}{ds^2}\right)^2\right\}\\ &\qquad\qquad+\rho\frac{d\mu}{ds}\left\{\frac{dx}{ds}\frac{d^2x}{ds^2}+\frac{dy}{ds}\frac{d^2y}{ds^2}+\frac{dz}{ds}\frac{d^2z}{ds^2}\right\}\\ &=\frac{\mu}{\rho}.\end{aligned}$$

Hence we get the equation

$$\frac{1}{\rho}=\frac{d}{dn}(\log\mu),$$

which is the differential equation of the path.

The Rainbow.

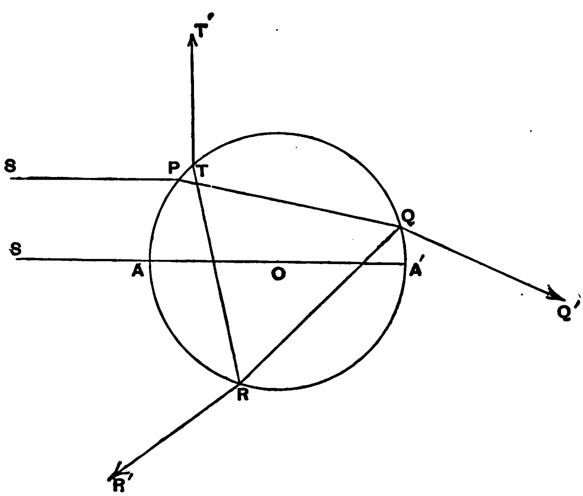
314. The first satisfactory explanation of the rainbow was given by Antonius de Dominis, archbishop of Spalatro, in a work

De Radiis Visus et Lucis, published in 1611. He shows that the inner bow is formed by two refractions and one intermediate reflexion of the sun's light in drops of rain; and the outer bow by two refractions and two intermediate reflexions. This explanation was adopted by Descartes and was confirmed by experiments made with glass globes filled with water, and arranged so as to exhibit the colours of the two bows. It remained for Newton to complete the theory by explaining the colours. The complete theory involves considerations which belong to Physical Optics and was effected by Sir G. Airy; we must confine ourselves to the approximate theory.

315. When the parallel rays of the sun strike a drop of water, part of the light will be scattered at the outer surface of the drop and serve to render the drop visible, and part will enter the drop by refraction; of those rays which enter the drop part will be refracted out of the drop at the incidence on the second surface of the drop, and part will be reflected back into the drop, and so on, for any number of incidences. Let us consider the rays which are incident in a plane of symmetry and which pass out of the drop by refraction after one internal reflexion; it is clear that they will not all emerge in the same direction, for the deviation will depend on the angle of incidence. Moreover, if the angle of incidence increase uniformly the deviation will vary sometimes rapidly, sometimes more slowly; and the more slowly the deviation changes the less will be the divergence of the emergent rays. If therefore the emergent rays be received on a screen, the band will not be uniformly bright, but will be brightest in those parts where the divergence is least, that is, where the deviation change most slowly. Now the changes of the deviation are slowest near a maximum or a minimum, and therefore at the spot where the deviation is a minimum the band will be much brighter than anywhere else. Within the direction of minimum deviation there will be no light transmitted.

If instead of a single drop, a shower of drops be illuminated by the rays of the sun, those drops whose positions are such that the rays emerge in the direction of the eye with minimum deviation will appear more brilliant than the others, and will be marked out against the cloud as specially bright. This phenomenon is the

same in all planes which pass through the line joining the sun and the observer's eye, and therefore the assemblage of bright drops will form an arc of a circle whose centre is on this line, and whose angular radius as seen by the eye only depends on the refractive index of the light. The refractive index is not the same for all the rays of a solar beam, being greatest for the violet and least for the red rays, and therefore the position of the bright arc will not be the same for all the coloured rays of the solar beam. There will therefore be a series of coloured bands exhibiting the colours of the solar spectrum. This is the principle of the explanation of the rainbow.



316. Let SP be a ray of light incident on the drop of water at P, PQ the ray refracted into the drop; part of the light will pass out by refraction at Q along the line QQ', while another part will be reflected at Q along the line QR, where part will pass out by refraction and part be reflected, and so on. Let ϕ be the angle of incidence at P, ϕ' the angle of refraction, so that $\sin\phi = \mu \sin\phi'$. The deviation at P is therefore $\phi - \phi'$. When the ray is incident at Q, the angle of incidence is ϕ'; and therefore for the part which passes out at Q a second deviation equal to $\phi - \phi'$ in the same direction as before is produced. But for the part reflected at Q, the deviation is $\pi - 2\phi'$, and where the ray meets the surface again at R the angle of incidence is again ϕ'. If therefore the ray under-

goes n internal reflexions and then passes out by refraction, the whole deviation will be

$$D = 2(\phi - \phi') + n(\pi - 2\phi').$$

The most efficacious rays, as we have seen, are those which make the deviation a maximum or minimum. To find the angle of incidence for these rays, we must equate to zero the first differential coefficient of D with regard to ϕ; we therefore get

$$0 = 1 - (n+1)\frac{d\phi'}{d\phi}.$$

From the equation, $\sin\phi = \mu \sin\phi'$, we find

$$\cos\phi = \mu\cos\phi' \frac{d\phi'}{d\phi},$$

and therefore, eliminating the differential coefficient,

$$\mu\cos\phi' = (n+1)\cos\phi.$$

Besides this we have the equation of refraction,

$$\mu\sin\phi' = \sin\phi;$$

squaring and adding both members of this equation, we get

$$\mu^2 = (n+1)^2\cos^2\phi + \sin^2\phi,$$

or

$$\cos\phi = \sqrt{\frac{\mu^2 - 1}{n^2 + 2n}}.$$

Since ϕ lies between 0 and $\frac{1}{2}\pi$, there is no ambiguity in this value of ϕ.

317. The value of μ for water is about $\frac{4}{3}$, and in order that the value of ϕ may be real, the numerator must be less than the denominator in the expression for $\cos\phi$; and therefore $(n+1)^2$ must be $> \mu^2$, or $(n+1)$ must be greater than $\frac{4}{3}$. Thus n must be equal to 1 at least, and the light emerging from the drop at Q does not possess either minimum or maximum deviation, and therefore forms no rainbow. There is no superior limit to the value of n, and, theoretically, bows may be formed after any number of internal reflexions.

318. So far we have not enquired whether the value of ϕ as

found above gives a maximum or minimum value of D or neither; we must examine the sign of the second differential coefficient.

By successive differentiation,

$$\frac{dD}{d\phi} = 2 - 2(n+1)\frac{d\phi'}{d\phi},$$

$$\frac{d^2D}{d\phi^2} = \quad -2(n+1)\frac{d^2\phi'}{d\phi^2}.$$

It was shown that

$$\frac{d\phi'}{d\phi} = \frac{\cos\phi}{\mu\cos\phi'};$$

from this we derive

$$\frac{d^2\phi'}{d\phi^2} = \frac{-\mu\cos\phi'\sin\phi + \mu\sin\phi'\cos\phi\dfrac{d\phi'}{d\phi}}{\mu^2\cos^2\phi'};$$

and therefore the sign of $\dfrac{d^2D}{d\phi^2}$ is the same as that of

$$\cos\phi'\sin\phi - \sin\phi'\cos\phi\frac{d\phi'}{d\phi}.$$

Substituting for $\dfrac{d\phi'}{d\phi}$ its value, this becomes

$$\frac{\mu\cos^2\phi'\sin\phi - \cos^2\phi\sin\phi'}{\mu\cos\phi'}.$$

Now, since ϕ' is always less than $\frac{1}{2}\pi$, the denominator of this fraction is positive, and the fraction takes the sign of

$$\mu(1-\sin^2\phi')\sin\phi - (1-\sin^2\phi')\sin\phi',$$

or

$$\mu\sin\phi - \frac{\sin\phi}{\mu};$$

that is, of

$$\sin\phi\frac{(\mu^2-1)}{\mu}.$$

In the case of raindrops, $\mu = \frac{4}{3}$, and therefore if we consider ϕ always positive, D will be a minimum for any number of internal reflexions. The deviation will of course be in different directions according as the incident ray falls on the upper or lower half of the drop.

319. We must next consider the order of the coloured rays by examining the changes in the direction of the most efficacious

rays for different refractive indices. If Δ denote this minimum deviation, Δ is determined by the equations

$$\Delta = n\pi + 2\phi - 2(n+1)\phi',$$

$$\mu \cos\phi' = (n+1)\cos\phi.$$

From the first of these equations we find

$$\frac{d\Delta}{d\mu} = 2\left\{\frac{d\phi}{d\mu} - (n+1)\frac{d\phi'}{d\mu}\right\}.$$

Also, differentiating the equation $\sin\phi = \mu \sin\phi'$,

$$\cos\phi\frac{d\phi}{d\mu} = \mu\cos\phi'\frac{d\phi'}{d\mu} + \sin\phi',$$

and therefore $$\frac{d\Delta}{d\mu} = \frac{2}{\cos\phi}\left[\sin\phi' + \{\mu\cos\phi' - (n+1)\cos\phi\}\frac{d\phi'}{d\mu}\right]$$

$$= \frac{2\sin\phi'}{\cos\phi};$$

that is, $$\frac{d\Delta}{d\mu} = \frac{2}{\mu}\tan\phi.$$

This shows that $\dfrac{d\Delta}{d\mu}$ is positive; therefore Δ increases with μ, and the minimum deviation is greatest for the violet rays and least for red rays.

320. It has been shown that in order to produce a rainbow, at least one reflexion inside the drop is necessary. At each subsequent reflexion part of the light will be lost, and the corresponding rainbows will be fainter. The rainbow produced by one internal reflexion is called the *primary rainbow*. The angle of incidence corresponding to the most efficacious rays is given by the formula,

$$\cos\phi = \sqrt{\frac{\mu^2-1}{3}},$$

and the deviation by the equation

$$D = 2(\phi - \phi') + \pi - 2\phi'.$$

The refractive indices of water for red and violet rays, respectively, are $\frac{108}{81}$ and $\frac{109}{81}$. If these values be substituted for μ in the preceding formula, we find by the aid of trigonometrical tables

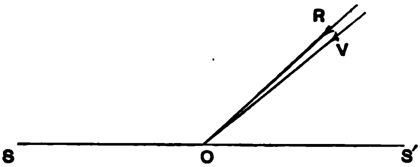
the values of the deviations corresponding to these rays to be,

$$D_R = 137^\circ\ 58'\ 20'',$$

$$D_V = 137^\circ\ 43'\ 20''.$$

Let O be the eye of the spectator, and SOS' a line drawn in the direction of the sun's rays; then, if we make the angle $S'OR$ equal to the supplement of D_R, that is, equal to $42^\circ\ 1'\ 40''$, RO will be the direction in which the most efficacious red rays will enter the eye. Similarily, if an angle $S'OV$ be constructed equal to the supplement of D_V, that is, equal to $40^\circ\ 16'\ 40''$, VO will be the direction in which the most efficacious violet rays will enter the eye, and the intermediate coloured rays will enter in directions intermediate between RO and VO.

And, further, if the lines OR, OV revolve round the line OS' as an axis, it is clear that all the drops on the conical surface



generated by the revolution of RO will transmit red rays copiously to the eye, and similarly for the other colours. To the eye therefore will appear a series of coloured arches with the violet rays innermost.

The effect of the rays which strike the eye with greater deviation, will be to light up the cloud within the bow with faint light, while no light will reach the eye from drops lying outside the bow.

The separation of the colours is not perfect, but they overlap each other, so that some of the colours can scarcely be recognised. The reason of this, just as in Newton's experiment with the prism, is that the sun has an angular diameter of 33′, and as each point of the sun sends out rays we get a series of rainbows due to the different elements of the sun's surface all superimposed and confused together.

There is yet another set of rays which pass through the drop with minimum deviation, those which strike the drop on its lower side at the same angle of incidence as before. These are directed

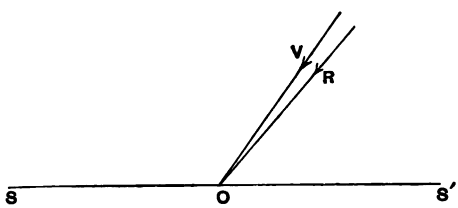
after refraction away from the earth, and are not seen by an observer on the earth; though they give bows which have sometimes been observed during balloon ascents, or on the summits of high mountains which lie above the clouds. When the sun is sufficiently near the horizon a complete circle may sometimes be seen in this manner.

321. When the rays undergo two internal reflexions they form a rainbow called the *secondary rainbow*. If we make $n = 2$, and substitute the same values of μ as before, we find

$$D_R = 230^\circ\ 58'\ 50'',$$

$$D_V = 234^\circ\ 9'\ 20''.$$

These deviations being greater than 180°, it is easy to see that the rays which reach the eye of an observer stationed on the earth are incident on the lower half of the drop.



Let SOS' be a line drawn through the observer's eye, in the direction of the sun's rays, and let angles $S'OR$, $S'OV$ be constructed, respectively equal to $D_R - 180^\circ$, $D_V - 180^\circ$, that is, to $50^\circ\ 58'\ 50''$, and $54^\circ\ 9'\ 20''$. Then RO, VO will be the directions of the most efficacious red and violet rays, respectively, and the phenomenon of the secondary rainbow may be deduced by revolving the lines OR, OV about the line OS' as before. The order of the colours is inverted in this bow, the violet being outside and the red inside. The rays which reach the eye with greater deviation serve to light up the cloud outside the bow. The secondary bow will be less bright than the primary bow, for two reasons; first, the light has undergone two internal reflexions and has thereby been weakened, and secondly, there is a greater angular dispersion of the rays in this rainbow than in the primary bow.

322. These two rainbows are the only ones which are usually

perceived, although the higher bows exist in theory. The third and fourth bows could never be seen except under special circumstances. For if we make $n = 3$, we find for red rays $D = 318^{\circ}\ 24'$ $= 360^{\circ} - 41^{\circ}\ 36'$. The direction of the rays will therefore pass behind the cloud, and to an observer stationed there it would be lost in the much brighter direct light from the sun.

If $n = 4$, $D = 360^{\circ} + 44^{\circ}\ 13'$. The case of four internal reflexions therefore differs little from the last; the efficacious rays will be incident on the upper half of the drop and will fall behind the cloud as before.

For the fifth arc, $D = 360^{\circ} + 126^{\circ}$, and the bow will have an angular radius of 54° and may be seen outside the secondary bow, especially in waterfalls where the drops are near the eye. The higher bows have never been seen except in laboratories under careful experimental conditions.

Halos and similar phenomena.

323. Besides the rainbow, which owes its existence to refractions and reflexions of the solar light by drops of water present in the air, there are other phenomena of a similar nature which are due to the presence in the air of ice-crystals, which reflect and refract the solar rays. These phenomena we now proceed to enumerate and explain.

The most frequent of them are called *halos;* these are coloured circles which appear round the sun, and sometimes also about the moon. The halo which is seen the most frequently has an angular radius of 22 degrees. The colours range from red inside to violet outside. This phenomenon, very common in northern regions, is not rare even in our climate; several of them are noted weekly at meteorological observatories.

Another circle, whose angular radius is 46 degrees, surrounds the former and presents the colours in the same order; this is called the halo of 46 degrees.

After these, the next phenomenon in order of frequency is a circle of white light passing through the sun parallel to the horizon. This is called the *parhelic circle.*

On the parhelic circle are seen several white or coloured images

of the sun; at the points where the circle meets the inner halo are two coloured images of the sun which are red on the inside. These images are clearly defined when the sun is on the horizon; when the sun has a greater altitude, they are seen a little outside the points of intersection. They are called *parhelia.*

More rarely are seen two similar images, situated also on the parhelic circle, at the points of intersection of that circle with the outer halo.

Rarer still are seen points on the parhelic circle which manifest a sudden increase of brightness. These points are not fixed; they are found between 90 and 140 degrees from the sun. They are called *paranthelia.* The *anthelion* is a white image which appears on the parhelic circle just opposite to the sun.

Outside the parhelic circle are sometimes found curves less simple than the halos and the parhelic circle. From the parhelia belonging to the inner halo there proceed two oblique arcs, called the *arcs of Löwitz.*

At other times, at the upper and lower parts of each halo, are seen *tangential arcs* which, for the inner halo, are occasionally prolonged and form a sort of elliptic halo; the halo of 46 degrees also has tangential arcs, but these arcs are never prolonged.

Finally, at the sides of the halo of 46° supra-lateral and infra-lateral tangential arcs are sometimes seen.

324. These phenomena cannot be explained by the action of small drops of water. Most of them are coloured and are therefore due to refractions. Also they appear in our climate more frequently in winter than in summer, and in northern countries they shine with an intensity and frequency unknown in our country.

Marriotte explained some of these appearances by the existence in the atmosphere of ice-crystals, and the others have been attributed to the same cause. Some of the assumptions are arbitrary, but Galle and Bravais have established the theory so as to leave little doubt of its validity.

The crystals of ice have been carefully observed and it is found that one crystalline form occurs more frequently than all others; this is the form of a hexagonal prism, which presents itself under two aspects, either much elongated like a needle, or very flat like a thin plate.

From these forms of the ice-crystals it follows that there will be three different refracting angles to consider. Two adjacent faces will be inclined at 120°, two faces not adjacent at 60°, and finally, the sides of the prism will form an angle of 90° with the base.

325. The halo of 22° was explained by Marriotte. If we suppose that the air contains prisms of ice distributed in all directions in space, there will always occur prisms whose edges will be perpendicular to the plane drawn through the sun and the observer's eye. Now the minimum deviation for a ray of light traversing a prism of ice whose refracting angle is 60° is exactly equal to 22°. It appears then that in all the directions which make an angle of 22° with the line joining the eye and the sun, there will be seen a maximum of light. Also, the angle of minimum deviation is smaller for red rays than for violet rays, and therefore it is clear that the halo will be coloured and will appear red inside and violet outside.

Cavendish explained the halo of 46°; he attributed it to the refraction of light across faces inclined to each other at 90 degrees; the minimum deviation for such a refraction is found by calculation to be 46°. This phenomenon is explained just as before, and the order of the colours is the same. But as the refracting angle is larger than for the inner halo, the refracted rays will be more scattered; it follows therefore that the halo of 46° is less luminous, for the light is spread over a ring of double the radius and double the breadth.

326. The two halos are the only phenomena which can be explained by supposing the ice-crystals to be distributed in all directions. But it will readily be imagined that certain directions will predominate; for the needle-shaped crystals, under the influence of their weight, will tend to assume a vertical position, while the flat crystals will direct themselves so that their bases are vertical.

The reflexion of light on the prisms of ice placed in all directions with their reflecting faces vertical causes the parhelic circle. If these vertical planes are very numerous they will produce on the eye an impression of a complete circle. The

reflexion at the vertical faces of the flat prisms will give rise to the same appearance. This explanation is due to Young.

327. The parhelia were explained by Marriotte. They are due to the presence in the air of vertical needle-shaped crystals. Suppose that there are a large number of vertical prisms whose refracting angles are 60°. If the sun be on the horizon the solar rays fall on the principal section of the prisms; the minimum deviation for such rays is equal to 22°, so that the parhelia are not only on the parhelic circle but on the halo of 22°. When the sun is above the horizon the solar rays are not in a principal plane; but when they emerge from the prisms they will all make the same angle with the refracting edges, that is with the vertical, as when they enter; so that the rays will appear to enter the eye from points on the parhelic circle. There will be a minimum of deviation in azimuth which will be greater than 22°, and which will depend on the altitude of the sun. As the minimum deviation will not be the same for different colours, it follows that the colours will form a spectrum, the red being nearest to the sun; further away from the sun the rays are superimposed so as to form a tail of white light which extends along the parhelic circle for a space of 10 to 20 degrees. The parhelia are more brilliant than the halos, because the vertical prisms are more numerous than those distributed in all directions.

The oblique arcs observed by Löwitz, have been explained by Galle and Bravais, as due to small oscillations of the vertical prisms about their vertical mean position; but the consequences of the theory have only been imperfectly verified by observations of the phenomena.

The parhelia of 46° are very rare, and their position is not very accurately known. M. Bravais regards them as produced at 44° by the parhelia of 22° which act like the sun.

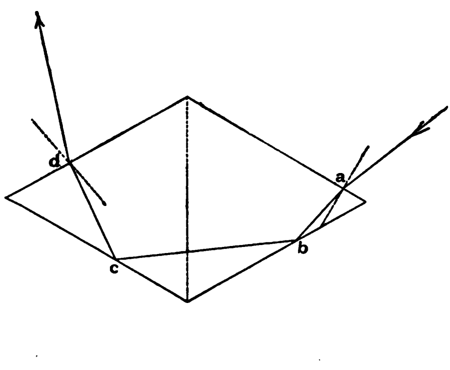
328. To explain the paranthelia, which are points on the parhelic circle which manifest a greater intensity of light than the rest of the circle, we must enquire into forms of prisms which will produce a constant deviation on rays of light. It is a well-known theorem that this may be brought about by two reflexions at plane surfaces; for if a ray of light be reflected at each of two

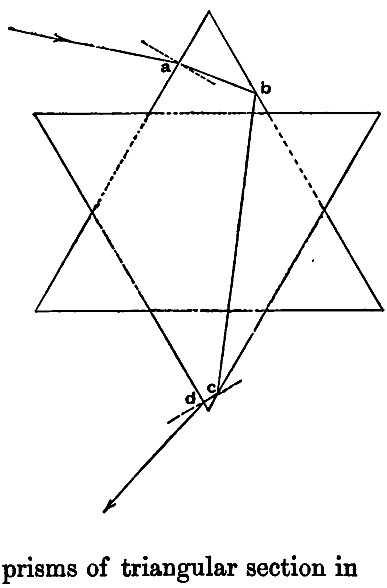
plane surfaces the deviation produced is equal to twice the angle between the reflecting surfaces. Now prisms of ice whose axes are vertical and such that two faces are in contact will present externally two reflecting faces inclined at 120° to each other. Rays reflected at these surfaces will be turned through an angle of 240°; this will give rise to two white images of the sun on the parhelic circle, each at an angular distance of 120° from the sun.

The same effect may be produced by reflexion at the interior faces of a prism. For when a ray enters across the face *a*, is reflected at the face *b*, and again at the face *c*, and finally emerges across the face *d*, if we call the three angles between these faces, (*ab*), (*bc*), (*cd*), it is easy to see that provided that

$$(ab) + (cd) = (bc)$$

the angles of incidence and emergence at the faces *a* and *d* are equal with opposite sign, and therefore the deviation is equal to twice the angle (*bc*), which is constant. The arrangement in question may be produced in several ways, by prisms of triangular or stellate section.

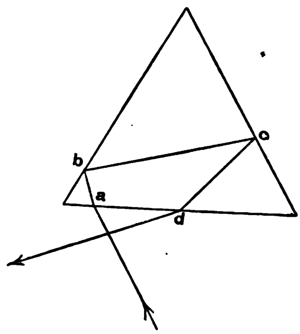




The first figure represents two prisms of triangular section in contact, and the other a stellate crystal with six points. In each case the deviation produced is 240°, and therefore the arrangement

will produce a white image of the sun on the parhelic circle at a distance of 120° from the sun.

Next, let us consider the path of a ray of light incident on a crystal whose right section is an equilateral triangle; we suppose the light to pass through in a principal plane, and that the ray is incident on the base, and is reflected by the two sides of the triangle in succession and finally emerges through the base. It is easy to see that the total deviation by this arrangement is equal to $\phi + \psi$, where ϕ and ψ are the angles of incidence and emergence.



This deviation is capable of a minimum situated in a direction making an angle of 98° with that of the solar rays; and there will be a coloured image of the sun on the parhelic circle, at about 98° from the sun.

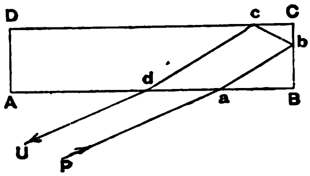
When the sun is not on the horizon the preceding investigations may be applied to the projections of the path of solar light on a horizontal plane, the refractive index in each case being altered to the ratio $\mu \cos \eta' : \cos \eta$, where η is the altitude of the sun, and

$$\sin \eta = \mu \sin \eta'.$$

329. The *anthelion* is a bright patch of white light with an ill-defined edge, often exceeding the apparent diameter of the sun, situated on the parhelic circle diametrically opposite to the sun. To explain this, it is necessary to suppose that the hexagonal lamellar prisms are disposed so that their axes are horizontal, and one of their three diagonals vertical. Consider the rays of light which after having traversed one of the four vertical faces of the

crystal are reflected twice inside the crystal at faces which are at right angles to each other and finally emerge by the face by which they entered. It is easy to see that the rays will emerge in a direction parallel to their original direction.

These reversed rays give rise to the anthelion. When the sun



is above the horizon the same argument may be applied to the horizontal projection of the path of the light.

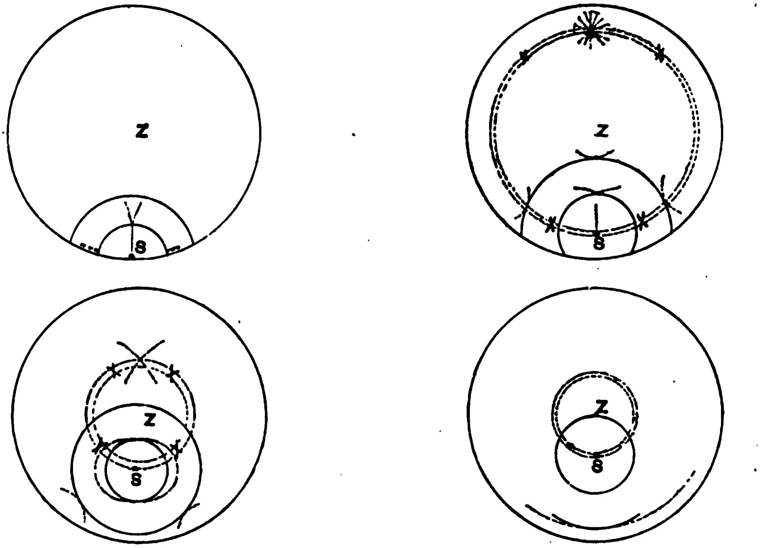
330. The *tangential arcs* were explained by Young as due to crystals whose refracting angles were 60° and whose axes are horizontal. If there are a very large number of these crystals with their axes in all possible horizontal directions, they will give rise to an infinite number of parhelia, of which one will be the parhelion of 22°, and will be situated at the highest point of the inner halo. This series of parhelia form two tangential arcs to the halo, and are sometimes united into a continuous curve. When there are a large number of these prisms with their axes horizontal, the sides of the prisms, being of small dimensions, will transmit only a small amount of light, so that the intensity of the arcs is very feeble compared with that of the parhelic circle.

The tangential arcs to the halo of 46° are seen more often and with greater brilliancy; they are due to the refraction of light through the refracting angles of 90°, which occur in the vertical prisms which are not pointed; these prisms are very frequent in the atmosphere. Each system of prisms whose edge is parallel to a particular direction in a horizontal plane gives rise to one point, and the series of these points form the tangential arc to the halo.

The lateral tangential arcs are due to the flat prisms whose axes are horizontal.

331. These are subjoined figures of the complete appearance of the halos for different altitudes of the sun; they are taken from M. Bravais' Memoir on Halos in the *Journal de l'École Royale*

Polytechnique, tome XVIII., where further details about the appearances of these phenomena and their theoretical explanation may be found.



EXAMPLES.

1. Prove that the power of a solid transparent spherical shell whose bounding radii are $\frac{1}{3}a\sqrt{3}$ and a, and in which the index of refraction at a distance r from the centre is $1 + r^2/a^2$, is $\pi/6a$.

2. A transparent sphere of radius a is such that the index of refraction at any point is $\mu\,(a/r - 1)^{\frac{1}{2}}$, where r is the distance from the centre. Prove that the path of every ray is an ellipse, and that all rays which emanate from a luminous point touch an ellipsoid of revolution.

3. Find the path of a ray through a medium in which μ varies as $\sqrt{(a^2+r^2)}/r$, and show that in a certain case the path is a reciprocal spiral.

4. A ray of light is propagated through a medium of variable density in a plane which divides the medium symmetrically; prove that the path is such that when described by a point with velocity always proportional to μ, the index of refraction, the accelerations of the point parallel to the (rectangular) axes of x and y will be proportional to $\frac{d(\mu^2)}{dx}$, $\frac{d(\mu^2)}{dy}$, respectively.

5. A ray is propagated through a medium of variable density in a plane (xy) which divides the medium symmetrically; prove that the projection of the radius of curvature at any point of the path of the ray on the normal to the surface of equal density through the point is $\mu \; : \; \sqrt{\left(\frac{d\mu}{dx}\right)^2 + \left(\frac{d\mu}{dy}\right)^2}$.

6. Show that in a refracting medium of index μ, the path of a ray in two dimensions, when referred to polar coordinates, can be put in the form

$$\frac{d\xi}{ds}-\eta\frac{d\theta}{ds}=\frac{d\mu}{dr}$$

$$\frac{d\eta}{ds}+\xi\frac{d\theta}{ds}=\frac{d\mu}{rd\theta},$$

where $\xi\equiv\mu\frac{dr}{ds}$, $\eta\equiv\mu r\frac{d\theta}{ds}$.

Of all the rays that can pass through a given point the one of minimum curvature there cuts orthogonally the refracting stratum at the point.

7. A ray passes through a medium of variable density whose refractive index μ varies as y^{-n}; show that the intrinsic equation to the path of the ray is

$$\frac{ds}{d\phi}=k\cos^{\frac{n-1}{n}}\phi.$$

8. If the refractive index of a medium at any point be proportional to its distance from a fixed plane, prove that the path of the ray will be the curve

$$\frac{2x}{a}=\frac{c}{a}e^{\frac{y}{a}}+\frac{a}{c}e^{-\frac{y}{a}},$$

a and c being constants.

9. A point of light is placed at the origin of coordinates in a medium where the refractive index is given by $\mu=e^{-kx}$. Show that an eye placed at the point (x, y) will see the origin of light by means of a small pencil of light, one of whose focal lines lies on the axis of x, and the other at a distance v from the eye, where

$$k^2v^2e^{kx}\cos^2 ky=2(\cosh kx-\cos ky),$$

while the axis of the pencil makes an angle ψ with the axis, where

$$\cot\psi\sin ky=\cos ky-e^{-kx}.$$

10. A medium is bounded by the planes of x and y, the refractive index at any point being given by the equation $\log\mu=xy/a^2$; two rays are incident on it parallel to the axes, respectively, and at equal distances c from the origin; show that if they intersect, it will be at an angle whose circular measure is

$$\frac{\pi}{2}-\frac{c^2}{a^2}.$$

11. The path of a ray through a medium of variable density is an arc of a circle in the plane of xy; prove that the refractive index at any point (x, y) must be $\frac{1}{x-a}f\left(\frac{x-a}{y-b}\right)$, where f is an arbitrary function, and (a, b) the centre of the circle.

12. If the path of a ray of light be $y=ae^{\frac{x}{c}}$, show that the index of refraction at any point is determined from the equation

$$\mu=(y^2+c^2)^{\frac{1}{2}}f\left(x+\frac{y^2}{c}\right).$$

13. If a small pencil of light pass directly through a plate of thickness b, the index of refraction being $f(x/c)$, x being measured from the plane of incidence, and c varying slightly with the colour of the light, show that the chromatic aberration on emergence is

$$\left\{\frac{b}{c}\frac{1}{f(b/c)} - \int_0^b \frac{dx}{cf(x/c)}\right\} dc;$$

$f(0/c)$ being supposed equal to unity.

14. A ray of light is incident perpendicularly on one of the faces of a prism whose density varies in such a manner that the coefficient of refraction at any point is μe^θ, where μ is constant, and θ the angle which a plane through the point and the edge of the prism makes with that face upon which the ray is incident. If α be the refracting angle of the prism, ϕ the angle of incidence on the second face, show that ϕ is determined by the equation,

$$\cos\phi - \sin\phi = e^{\phi - 2\alpha}.$$

15. The density of a prism at any point varies as its distance from the nearest face of the prism. If a ray pass through it in a principal plane, its distance from the edge at the point of incidence and emergence being a, and its nearest approach to the edge being c, show that the deviation is given by the equation,

$$\sin\left(\beta + \frac{D}{2}\right) = \sin\beta\, e^{\frac{Dc\sin\beta}{2(a - c\cos\beta)}},$$

where 2β is the vertical angle of the prism.

16. A pencil diverging from a point and originally a quadric cone passes through a heterogeneous medium; show (1) that the section of the pencil made by a plane perpendicular to the axis of the pencil is an ellipse whose axes do not necessarily lie in the principal planes of the pencil; (2) that there will be no *circle* of least confusion of the pencil, but that the ratio of the axes of any section will be least when the section is made at a distance h from the face, where

$$\frac{h - \rho_2}{h - \rho_1} = \frac{\rho_2}{\rho_1}\beta,$$

ρ_1, ρ_2 being the distances of the foci from the face, and β the ratio of the axes of the face which lie in the principal planes.

Show also that the greatest section in area between the focal lines is in every case at a distance $\frac{1}{2}(\rho_1 + \rho_2)$ from the front.

17. If b, b' be the breadths of the pth and qth rainbows, respectively, and δ the sun's apparent diameter, show that

$$b' - b = \left[\left\{\frac{(q+1)^2 - \mu^2}{(p+1)^2 - \mu^2}\right\}^{\frac{1}{2}} - 1\right](b - \delta), \text{ nearly.}$$

18. Show that in the theory of the primary rainbow, the caustic of the emergent pencil (after one internal reflexion) has two sheets, one of which terminates in a cusp and the other abruptly in a circle; and find the envelope of all asymptotes common to the two sheets. If ϕ be the angle of refraction of the ray which passes with minimum deviation, and 2ψ the deviation of this ray, show that

$$3\tan^2\psi + 1 = 2\tan^3\phi\tan\psi.$$

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